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Stabilité des images inverses des fibrés tangents et involutions des variétés symplectiques

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Table des matières

Chapitre 1

Introduction

Dans cette thèse j'ai travaillé sur deux problèmes différents dans la domaine de la Géométrie Algébrique.

Dans la première partie de ce travail, formée par les trois premiers chapitres, on étude la stabilité des images inverses du fibré tangent de l'espace projectif sur les courbes et après sur les surfaces. Ce sujet avait déjà fait l'objet de travaux de Paranjape et Ramanan [**29**], Ein et Lazarsfeld [**14**] et Beauville [**6**]. Notre point de départ est la thèse de Paranjape [**28**].

Dans le Chapitre 2 on donne les premières définitions et propriétés des tels fibrés. Soit L un fibré en droites engendré par ses sections globales sur une courbe projective lisse C de genre $g \geq 2$ sur un corps k algébriquement clos ; le fibré M_L est défini par la suite exacte courte

$$0 \longrightarrow M_L \longrightarrow H^0(X,L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

et on appelle E_L le dual de M_L. Comme L est engendré par ses sections globales, il définit un morphisme $\phi_L : C \longrightarrow \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$; en faisant le pull-back par ϕ_L de la suite exacte d'Euler (2.2.4) et le produit tensoriel par L^*, on obtient $E_L = \phi_L^* T_{\mathbb{P}^r} \otimes L^*$, donc la stabilité de E_L est équivalente à celle de $\phi_L^* T_{\mathbb{P}^r}$.

On rappelle les résultats déjà connus dans la littérature et on remarque aussi qu'en dimension 2 les fibrés de la forme E_L sont simples et rigides, deux propriétés plus faibles que la $H-$stabilité.

Dans le Chapitre 3 on commence par rappeler la définition des indices de Clifford $c(L)$ et $c(C)$. Dans [**28**] Paranjape montre la proposition suivante.

Proposition 1.1. *Si $c(C) \geq c(L)$ alors E_L est semi-stable. Si $h^1(C,L) = 1$ et si on a $c(C) > 0$ ou $c(C) > c(L)$ alors E_L est aussi stable.*

En complétant sa démonstration on obtient alors le théorème suivant.

Théorème 1.2. *Soit L un fibré en droites sur une courbe projective lisse C de genre $g \geq 2$ engendré par ses sections globales tel que $\deg L \geq 2g - c(C)$. Alors :*

(1) *E_L est semi-stable ;*

(2) *E_L est stable sauf si $\deg L = 2g$ et C est hyperelliptique ou $L \cong K_C(p+q)$ avec $p, q \in C$.*

Le Théorème 1.2 est le meilleur énoncé qu'on peut obtenir si on cherche un résultat valable pour toute courbe en donnant seulement des conditions sur le degré. En effet, on construit des fibrés L avec $\deg L = 2g - c(C) - 1$ tels que E_L ne soit pas semi-stable.

Proposition 1.3. *Soit C une courbe $d-$gonale de genre $g \geq 2$ et d'indice de Clifford $c(C) = d - 2 < \frac{g-2}{2}$; il existe un fibré en droites L de degré $\deg L = 2g - c(C) - 1$ sur C engendré par ses sections globales et non-spécial tel que E_L ne soit pas semi-stable.*

Dans le cas des courbes hyperelliptiques on arrive à caractériser complétement la stabilité de E_L.

Proposition 1.4. *Soit L un fibré en droites sur une courbe projective lisse C hyperelliptique de genre $g \geq 2$ engendré par ses sections globales tel que $h^0(C,L) \geq 3$ et soit $H = \mathcal{O}_C(\mathfrak{g}^1_2)$. Alors :*

(1) *E_L est stable si et seulement si $\deg L \geq 2g+1$;*

(2) *E_L est semi-stable si et seulement si $\deg L \geq 2g$ ou s'il existe un entier $k > 0$ tel que $L = H^{\otimes k}$.*

Dans le Chapitre 4 on regarde le même problème dans le cas des surfaces projectives lisses : comme la $H-$stabilité d'un fibré vectoriel E, où H est un fibré en droites engendré par ses sections globales sur X, est par définition la stabilité de sa restriction $E_{|C}$ à la courbe $C \in |H|$, c'est naturel d'essayer d'utiliser les résultats obtenus pour les courbes pour étudier le même problème sur les surfaces.

Dans la Section 4.2 on obtient quelques résultats sur les surfaces régulières, parmi lesquels le suivant

THÉORÈME 1.5. *Soient X une surface K3 projective lisse sur $\mathbb{C}$ et L un fibré en droites ample engendré par ses sections globales sur X ; le fibré vectoriel E_L est $L-$stable.*

Dans la Section 4.3 on étude le cas des surfaces abéliennes, en démontrant l'énoncé suivant

THÉORÈME 1.6. *Soient X une surface abélienne projective lisse sur $\mathbb{C}$ et L un fibré en droites engendré par ses sections globales sur X tel que $L^2 \geq 14$; le fibré vectoriel E_L est $L-$stable.*

Dans la deuxième partie de la thèse, formée par le Chapitre 4, on étude les involutions des variétés irréductibles holomorphes symplectiques, en particulier les involutions symplectiques. Ce type d'involutions et plus en général d'automorphismes d'ordre fini sur les surfaces K3 a été étudié pour la première fois par Nikulin dans [**26**]. Comme les variétés irréductibles holomorphes symplectiques sont la généralisation naturelle de surfaces K3 en dimension plus haute, Beauville a commencé l'étude de ce problème pour ces variétés dans [**4**]. Parmi la littérature sur ce sujet, qui contient différents points de vue, on rappelle ici les travaux de Boissière [**9**] et Boissière-Sarti [**10**] sur les involutions du schéma d'Hilbert d'une surface K3 et l'article de Beauville [**7**] où il étude le même problème pour les involutions antisymplectiques.

Dans les premières sections du Chapitre 5 on rappelle la définition et les propriétés des variétés irréductibles holomorphes symplectiques et l'outil principal qu'on utilise, la formule de Lefschetz holomorphe introduite par Atyah-Singer dans [**2**].

Dans la Section 5.4 on remarque que les composantes irréductibles du lieu fixe d'une involution symplectique i sont des sous-variétés symplectiques lisses et donc si X est une variété irréductible holomorphe de dimension 4 les composantes du lieu fixe de i sont soit des points fixes isolés soit des surface lisses K3 ou abéliennes.

Dans la Section 5.5 on démontre le résultat principal de cette partie,

THÉORÈME 1.7. *Soient X une variété irréductible holomorphe symplectique de dimension 4 telle que $b_2(X) = 23$ et i une involution symplectique de X. Soient τ la trace de i^* sur $H^{1,1}(X)$, N et K respectivement le nombre des points fixes isolés et de surfaces K3 des points fixes. Seuls les cas suivants sont possibles :*

(1) $\tau = -3$, $N = 12$ *et* $K = 0$;
(2) $\tau = 3$, $N = 36$ *et* $K = 0$;
(3) $\tau = 5$, $N = 28$ *et* $K = 1$.

De plus dans les deux premiers cas i fixe au moins une surface abélienne.

On conjecture qu'une involution symplectique ne fixe jamais une surface abélienne, donc le seul cas possible serait le troisième, avec 28 points fixes isolés et une surface K3 fixée. Les sections suivantes du Chapitre 5 sont dédiées à fournir des arguments en faveur de cette conjecture, en montrant que dans les exemples connus les involutions symplectiques la vérifient. On regarde donc les involutions naturelles sur le schéma d'Hilbert d'une surface K3 dans la Section 5.6, les involutions de la variété de Fano d'une cubique lisse de $\mathbb{P}^5$ induites par une involution de $\mathbb{P}^5$ dans la Section 5.7 et les involutions de la variété recouvrement double d'une sextique EPW induites par une involution de $\mathbb{P}^5$ dans la Section 5.8.

Les chapitres qui suivent sont les articles en anglais qui contiennent les résultats de cette thèse.

Chapitre 2

Inverse images of the tangent bundle of $\mathbb{P}^r$

Résumé. Soit L un fibré en droites engendré par ses sections globales sur une variété projective lisse X sur un corps k algébriquement clos. Le fibré L définit $\phi_L\colon X \to \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$ et $\phi_L^* T_{\mathbb{P}^r}$ est l'image inverse sur la variété X du fibré tangent de $\mathbb{P}^r$. Après avoir donné les premières définitions et propriétés des tels fibrés, on rappelle les résultats déjà connus dans la littérature, en expliquant en details les techniques utilisées par Ein et Lazarsfeld dans [**14**], par Paranjape dans [**28**] et par Beauville dans [**6**]. On remarque aussi qu'en dimension 2 les fibrés E_L sont simples et rigides, deux propriétés plus faibles que la $H-$stabilité.

2.1. $H-$Stability

For an accurate and complete survey on this subject we refer the reader to Chapter 1 of [**22**] ; here we limit ourselves to recall the basic definitions and properties that we will need later.

Definition. *Let X be a smooth projective variety of dimension n over an algebraically closed field k and H a line bundle on X generated by its global sections ; a vector bundle E of rank r is said to be $H-$stable [$H-$semistable] if all its quotient sheaves F of rank $0 < \operatorname{rk} F < r$ satisfy $\mu(F) > \mu(E)$ [resp. $\geq$], where $\mu(F) = \frac{c_1(F).H^{n-1}}{\operatorname{rk} F}$ is the ($H-$)slope of F.*

One can also give an equivalent definition using subsheaves instead of quotients and requiring the opposite slope inequality. Moreover it turns out that it is enough to verify the definition for quotient bundles in the case of curves and for quotient sheaves without torsion in the case of surfaces.

When $n = 1$ the slope becomes $\mu(F) = \frac{d}{r}$, hence we recover the usual definition given in the case of curves.

Let us recall a few properties satisfied by $H-$stable vector bundles.

– if E is $H-$stable then E is $H-$semistable ;
– a line bundle L is always $H-$stable for any choice of H ;
– for any line bundle $L \in \operatorname{Pic}(X)$, E is $H-$stable if and only if $E \otimes L$ is ;
– E is $H-$stable if and only if E is $aH-$stable for any integer $a > 0$.

As a first example of $H-$stable vector bundle let us consider the tangent bundle of $\mathbb{P}^n$.

Lemma 2.1. *The vector bundle $T_{\mathbb{P}^n}$ is $\mathcal{O}_{\mathbb{P}^n}(1)-$stable for all integers n.*

Proof. See for example [**22**] Lemma 1.4.5. □

Lemma 2.2. *Let E and F be two semistable vector bundles on a variety X. Then if $Hom(E, F) \neq 0$, $\mu(E) \leq \mu(F)$. If E is stable, F is semistable and $\mu(E) = \mu(F)$, any nontrivial homomorphism $f : E \to F$ is injective. If E and F are stable vector bundles and $\mu(E) = \mu(F)$, any non trivial homomorphism $f : E \to F$ is an isomorphism.*

Proof. The proof is analogous to that of Proposition 1.2.7 in [**22**] . □

2.2. First definitions

Given a line bundle L generated by its global sections on a smooth projective variety X over an algebraically closed field k, one can consider the kernel of the evaluation map

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0 \tag{2.2.1}$$

and its dual $E_L = M_L^*$.

Since the vector bundle L is spanned, the evaluation map is surjective and has constant rank, hence M_L and its dual are both vector bundles on X. An easy computation shows that $\operatorname{rk} M_L = \operatorname{rk} E_L = h^0(L) - 1$ and that $\det E_L = L$.

Moreover the vector bundle E_L is generated by its global sections : indeed, the dual exact sequence of (2.2.1)

$$0 \longrightarrow L^* \longrightarrow H^0(X,L)^* \otimes \mathcal{O}_X \longrightarrow E_L \longrightarrow 0 \tag{2.2.2}$$

shows that E_L is the quotient of a trivial vector bundle.

Another important property of M_L comes from the long exact cohomology sequence associated to (2.2.1)

$$0 \longrightarrow H^0(M_L) \longrightarrow H^0(L) \longrightarrow H^0(L) \longrightarrow H^1(M_L) \longrightarrow \cdots \tag{2.2.3}$$

Since the evaluation map induces isomorphism on global sections, we have $H^0(M_L) = 0$.

REMARK 2.3. *The last two properties of these bundles will turn out to be crucial to study their stability since they are inherited respectively by quotient sheaves of E_L and by subsheaves of M_L. Indeed, the cohomology sequence associated to (2.2.1) shows that $H^0(M_L) = 0$, hence for all subsheaves N of M_L we have $H^0(N) = 0$; in particular $\mathcal{O}_X$ cannot be a subbundle of M_L.*

Let us briefly recall the geometric interpretation of E_L : since L is generated by its global sections, the morphism $\phi_L : X \longrightarrow \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$ is well-defined and we have $L = \phi_L^* \mathcal{O}_{\mathbb{P}^r}(1)$; thus, from the dual sequence of (2.2.1) and from the well-known Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow H^0(X,L)^* \otimes \mathcal{O}_{\mathbb{P}^r}(1) \longrightarrow T_{\mathbb{P}^r} \longrightarrow 0 \tag{2.2.4}$$

it follows that $E_L = \phi_L^* T_{\mathbb{P}^r} \otimes L^*$; the $H-$stability of E_L is equivalent to that of $\phi_L^* T_{\mathbb{P}^r}$.

2.3. Different approaches to the problem

The $H-$stability of vector bundles E_L has been studied in the case of a curve by Paranjape in [**29**] with Ramanan and in his Ph.D. thesis [**28**] ; in particular, the latter contains the statements on which all our results in [**11**] and in [**12**] rely. Later Ein and Lazarsfeld showed in [**14**] that M_L is stable if $\deg L > 2g$ and Beauville investigated the case of degree $2g$ in [**6**]. Let us review briefly the different approaches to the problem contained in these papers.

Paranjape's approach. Let us postpone a more detailed account of his work to Chapter 3, since his Ph.D. thesis and the results inside are the starting point for our analysis in the case of curves. Here we just want to mention that, using an important numerical invariant of a curve, i.e. its Clifford index $c(C)$, Paranjape obtains the following theorem

THEOREM 2.4. *If $c(C) \geq c(L)$ then E_L is semistable ; moreover, if $h^1(L) = 1$ and either $c(C) > 0$ or $c(C) > c(L)$ the vector bundle E_L is also stable.*

In the case of the canonical bundle $L = K_C$, Paranjape and Ramanan in [**29**] prove a better result

THEOREM 2.5. *E_{K_C} is semistable and also stable if C is not hyperelliptic.*

These results are then used to study some conjectures by Green.

Ein and Lazarsfeld's approach. In their paper [**14**] they study a stronger property than the stability of E_L, the so-called cohomological stability.

DEFINITION. *A vector bundle E of rank r on a smooth irreducible projective curve C of genus $g \geq 1$ is said to be cohomologically stable [semistable] if for all $t < \operatorname{rk} E$ we have $H^0(\wedge^t E \otimes A) = 0$ for all line bundles A of degree $a \leq -t\mu(E)$ [resp. $a < -t\mu(E)$].*

It is easily shown that cohomological stability implies stability in the slope sense.

LEMMA 2.6. *If E is cohomologically stable [resp. semistable], it is also stable [semistable].*

Proof. Let $T \subsetneq E$ be such that $\deg T = a$ and $\mathrm{rk}\, T = t$; let us consider $A^* := \wedge^t T$, which is of degree a. There is an inclusion $A^* \subset \wedge^t E$ and hence there is a section $s \in H^0(\wedge^t E \otimes A) \setminus \{0\}$. Such a section s cannot exist if $\deg A = -a \leq -t\mu(E)$, i.e. $\mu(T) \geq \mu(E)$. Hence $\mu(T) < \mu(E)$. □

Ein and Lazarsfeld then show the following

THEOREM 2.7. *If* $\deg L \geq 2g+1$ *[*$\geq 2g$*],* E_L *is cohomologically stable [cohomologically semistable].*

In order to sketch the proof we need the following lemma, shown by Lazarsfeld in [**23**].

LEMMA 2.8. *Let us consider* $r = r(L) = d - g$ *and* $x_1, \ldots, x_{r-1} \in C$ *points such that* $L(-D)$ *is generated by its global sections and such that* $H^1(L(-D)) = 0$*, where* $D = \Sigma x_i$ *; then there is the following short exact sequence*

$$0 \longrightarrow L^*(D) \longrightarrow M_L \longrightarrow \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-x_i) \longrightarrow 0$$

Proof. Let us consider the short exact sequence

$$0 \longrightarrow L(-D) \longrightarrow L \longrightarrow L_{|D} \longrightarrow 0 \tag{2.3.1}$$

where $L_{|D}$ is a torsion sheaf of degree $\deg D = r - 1$ and $L(-D)$ has degree $\deg L - \deg D = g + 1$. Since the evaluation map is surjective, we get the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_{L(-D)} \arrow[r] \arrow[d] & H^0(L(-D)) \otimes \mathcal{O}_C \arrow[r] \arrow[d] & L(-D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_L \arrow[r] \arrow[d] & H^0(L) \otimes \mathcal{O}_C \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{M}_L \arrow[r] \arrow[d] & W_D \otimes \mathcal{O}_C \arrow[r, "u_D"] \arrow[d] & L_{|D} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $W_D = H^0(L)/H^0(L(-D))$. Since both the second and the third column are exact, it follows from the snake lemma that also the first column is exact.

Since $h^1(L(-D)) = 0$, from the cohomology sequence associated to (2.3.1) it follows that

$$h^0(L(-D)) = h^0(L) - h^0(L_{|D}) = r + 1 - (r - 1) = 2.$$

So $\mathrm{rk}\,(H^0(L(-D)) \otimes \mathcal{O}_C) = 2$ and $\mathrm{rk}\, M_{L(-D)} = 1$, hence $M_{L(-D)} = \det M_{L(-D)} = L^*(D)$.

Moreover, since $L(-D)$ is non-special, from the following cohomology sequence

$$0 \longrightarrow H^0(L(-D)) \longrightarrow H^0(L) \longrightarrow H^0(L_{|D}) \longrightarrow 0$$

we get $W_D \cong H^0(L_{|D}) \cong \bigoplus_{i=1}^{r-1} L_{x_i}$. Then $u_D = \oplus u_{x_i}$, where for all $i = 1, \ldots, r - 1$ the map u_{x_i} satisfies

$$0 \longrightarrow \mathcal{O}_C(-x_i) \longrightarrow \mathcal{O}_C \xrightarrow{u_{x_i}} \mathcal{O}_{x_i} \longrightarrow 0$$

It follows that the kernel is $\ker u_D = \oplus \ker u_{x_i}$, i.e. $\bar{M}_L = \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-x_i)$. □

REMARK 2.9. *If the points* $x_1, \ldots, x_{r-1}$ *are generic, the hypotheses of Lemma (2.8) are satisfied for* $\deg L \geq 2g + 1$*.*

Proof of Theorem 2.7. Let A be a line bundle of degree a such that, given $t < \operatorname{rk} M_L = d - g$, we have $a \leq -t\mu(M_L) = t\frac{d}{d-g}$. By Lemma 2.8 we have the following exact sequence

$$0 \longrightarrow L^*(D) \longrightarrow M_L \longrightarrow \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-x_i) \longrightarrow 0$$

with $x_1, \ldots, x_{r-1} \in C$ general points and $\operatorname{rk} L^*(D) = 1$. Hence we have the short exact sequence

$$0 \longrightarrow \wedge^{t-1}(\bigoplus \mathcal{O}(-x_i)) \otimes L^*(D) \longrightarrow \wedge^t M_L \longrightarrow \wedge^t(\bigoplus \mathcal{O}(-x_i)) \longrightarrow 0$$

i.e.

$$0 \longrightarrow \bigoplus L^*(D - x_{i_1} - \cdots - x_{i_{t-1}}) \longrightarrow \wedge^t M_L \longrightarrow \bigoplus \mathcal{O}(-x_{i_1} - \cdots - x_{i_t}) \longrightarrow 0$$

Tensoring with A one then gets a short exact sequence

$$0 \longrightarrow \bigoplus L^*(D - x_{i_1} - \ldots - x_{i_{t-1}}) \otimes A \longrightarrow \wedge^t M_L \otimes A \longrightarrow \bigoplus \mathcal{O}(-x_{i_1} - \ldots - x_{i_t}) \otimes A \longrightarrow 0$$

In cohomology this gives an exact sequence

$$0 \longrightarrow H^0(\bigoplus L^*(D - x_{i_1} - \cdots - x_{i_{t-1}}) \otimes A) \longrightarrow H^0(\wedge^t M_L \otimes A) \longrightarrow$$

$$\longrightarrow \bigoplus H^0(A(-x_{i_1} - \cdots - x_{i_t})) \longrightarrow \cdots$$

An easy computation shows that $\deg(D - x_{i_1} - \cdots - x_{i_{t-1}}) = r - t$ and $\deg(-x_{i_1} - \cdots - x_{i_t}) = -t$. Let D_t be a general effective divisor of degree t and D_{r-t} a general effective divisor of degree $r - t$. We need to show that

(1) $H^0(A(-D_t)) = 0$,

(2) $H^0(A \otimes L^*(D_{r-t})) = 0$,

to get $H^0(\wedge^t M_L \otimes A) = 0$. Since $a \leq -t\mu(M_L)$, we have :

(1) $\deg A(-D_t) = a - t$;

(2) $\deg(A \otimes L^*(D_{r-t})) = a - t - g$.

Moreover $d \geq 2g + 1$ implies that

$$\frac{a}{t} \leq -\mu(M_L) = 1 + \frac{g}{d-g} < 2.$$

Then from $t < d - g$ it follows that $a - t < g$. Hence $H^0(A(-D_t)) = 0$, because D_t is general and effective : indeed, if $H^0(A(-D_t)) \neq 0$ then $E = A(-D_t)$ is effective of degree $a - t$, but

$$\dim J = g > a - t = \dim \left\{E \in J / \deg E = a - t, h^0(E) \neq 0\right\}$$

hence for a general D_t we have $H^0(A(-D_t)) = 0$. From $a - t < g$ we also get that $\deg(A \otimes L^*(D_{r-t})) < 0$, so $H^0(A \otimes L^*(D_{r-t})) = 0$. $\square$

Ein and Lazarsfeld then use this and other results to investigate the Θ_C−stability of the degree d Picard bundle over $J_d(C)$, which they prove for $d \geq 2g$.

Beauville's approach. Using some properties of the theta divisor Beauville in [**6**] shows

THEOREM 2.10. *If $\deg L = 2g$, L is very ample and C is not hyperelliptic, E_L is stable.*

DEFINITION. *Let C be a curve of genus g and E a vector bundle on C of rank r and slope $\mu \in \mathbb{Z}$; let J^ν be the translated Jacobian of C that parametrizes line bundles of degree $\nu = g - 1 - \mu$ on C. We say that E admits a theta divisor if $H^0(E \otimes L) = 0$ for L general in J^ν ; in this case there is an effective divisor*

$$\Theta_E = \{L \in J^\nu / H^0(E \otimes L) \neq 0\},$$

called theta divisor.

If E admits a theta divisor, E is semistable : otherwise there would be a subbundle $F \subset E$ of slope $> \mu$ such that, by Riemann-Roch theorem, $H^0(F \otimes L) \neq 0$ for all $L \in J^\nu$ and this would imply $H^0(E \otimes L) \neq 0$ for all $L \in J^\nu$.

If E is semistable but not stable, its theta divisor (if it exists) is reducible : more precisely, there is a subbundle $F \subset E$ of slope μ and we have $\Theta_E = \Theta_F + \Theta_{E/F}$.

Theorem 2.11. *Let C be a non-hyperelliptic curve of genus g and L a line bundle of degree $2g$ on C generated by its global sections. Then :*

(1) *E_L admits a theta divisor,*

$$\Theta_{E_L} = (C_{g-2} - C) + \Theta_{L \otimes K_C^*}$$

in J^{g-3}, where C_d is the locus of effective divisors of degree d in J^d.

(2) *E_L is semistable ; it is stable if and only if L is very ample.*

Proof. Let us compute the theta divisor

$$\Theta_{M_L} = \{P \in J^{g+1} / H^0(C, M_L \otimes P) \neq 0\}.$$

Tensoring with P the sequence (2.2.1) in cohomology we get an exact sequence

$$0 \longrightarrow H^0(C, M_L \otimes P) \longrightarrow H^0(C, L) \otimes H^0(C, P) \xrightarrow{m} H^0(C, L \otimes P) \longrightarrow \cdots$$

Hence the theta divisor is the set of line bundles $P \in J^{g+1}$ such that m is not injective.

If $h^0(C, P) > 2$ then we have $\dim H^0(C, L) \otimes H^0(C, P) > h^0(C, L \otimes P)$ and $P \in \Theta_{M_L}$. If $h^0(C, P) = 2$ and $|P|$ has a base point, then $\dim H^0(C, L) \otimes H^0(C, P) = h^0(C, L \otimes P) = 2g + 2$; hence, if m is injective it is also surjective and the linear system $|L \otimes P|$ has a base point, impossible since $\deg L \otimes P = 3g + 1$. So $P \in \Theta_{M_L}$. Finally, if $h^0(C, P) = 2$ and the linear system $|P|$ is base points free, there is an exact sequence

$$0 \longrightarrow P^* \longrightarrow H^0(C, P) \otimes \mathcal{O}_C \longrightarrow P \longrightarrow 0$$

and in cohomology we get the induced exact sequence

$$0 \longrightarrow H^0(C, L \otimes P^*) \longrightarrow H^0(C, L) \otimes H^0(C, P) \xrightarrow{m} H^0(C, L \otimes P) \longrightarrow \cdots$$

hence m is not injective if and only if $H^0(C, L \otimes P^*) \neq 0$.

The line bundles of the two first cases have exactly the form $P'(q)$, with $q \in C$ and $P' \in J^g$ such that $h^0(C, P') \geq 2$; the ones in the last case are of the form $L \otimes Q^*$, with $Q \in \Theta_{\mathcal{O}_C} \subset J^{g-1}$. Since Θ_{E_L} is the image of Θ_{M_L} via the isomorphism $P \in J^{g+1} \mapsto K_C \otimes P^* \in J^{g-3}$, we get $\Theta_{E_L} = (C_{g-2} - C) \cup \Theta_{L \otimes K_C^*}$ as sets.

It is known that $C_{g-2} - C$ is irreducible with cohomology class $(g-1)\theta$ (see [**16**]). Since Θ_{E_L} has cohomology class $g\theta$, we get the statement.

Since E_L admits a theta divisor, it is semistable. If E_L is not stable, its stable components are $L' = L \otimes K_C^*$ and a vector bundle of rank $g - 1$. Let us remark that L' cannot be a quotient of E_L, since it is not spanned. Moreover, from the exact sequence

$$0 \longrightarrow L^* \otimes L'^* \longrightarrow H^0(C, L)^* \otimes L'^* \longrightarrow E_L \otimes L'^* \longrightarrow 0$$

it follows that, since $L^* \otimes L'^* = K_C \otimes L^{-2}$, in cohomology we obtain the following exact sequence

$$\cdots \longrightarrow H^0(E_L \otimes L'^*) \longrightarrow H^1(K_C \otimes L^{-2}) \xrightarrow{m^*} H^0(L)^* \otimes H^1(K_C \otimes L^*) \longrightarrow \cdots$$

As a consequence, $H^0(C, E_L \otimes L'^*) = 0$ if and only if m^* is injective, if and only if $m : H^0(C, L) \otimes H^0(C, L) \longrightarrow H^0(C, L^{\otimes 2})$ is surjective. From a theorem by Green and Lazarsfeld in [**18**] we know that m is surjective if and only if L is very ample. $\square$

In his paper then Beauville applies this statement to construct an example of stable vector bundles with reducible theta divisor.

2.4. Simplicity and rigidity of E_L

Let us first of all underline that the bundles E_L satisfy in almost any case a less strong property, the simplicity.

DEFINITION. *A vector bundle E on a smooth projective variety X over a field k is simple if* $\mathrm{End}(E) \cong k$.

This is a weaker property than $H-$stability.

LEMMA 2.12. *If k is an algebraically closed field, all stable sheaves are simple, i.e. their endomorphisms are scalar multiples of the identity.*

Proof. For every coherent sheaf F, the algebra $\mathrm{End}(F)$ has finite dimension over k. If E is a stable sheaf, then $\mathrm{End}(E)$ is an extension field of k. As we are supposing that k is algebraically closed, then $\mathrm{End}(E) = k$. □

PROPOSITION 2.13. *Let X be a smooth complex projective variety and let L be a big line bundle generated by its global sections on X ; if $\dim X \geq 2$, E_L is simple.*

Proof. If we tensor with E_L the short exact sequence (2.2.1) in cohomology we get an exact sequence

$$0 \longrightarrow H^0(M_L \otimes E_L) \longrightarrow H^0(L) \otimes H^0(E_L) \xrightarrow{\alpha} H^0(L \otimes E_L) \longrightarrow \tag{2.4.1}$$

$$\longrightarrow H^1(M_L \otimes E_L) \longrightarrow H^0(L) \otimes H^1(E_L) \longrightarrow \cdots$$

Since $H^0(L^*) \cong H^1(L^*) \cong 0$ by Ramanujan-Kodaira vanishing theorem (see [**25**]), we also have $H^0(L)^* \cong H^0(E_L)$. Now, by tensoring the dual sequence of (2.2.1) with L we obtain in cohomology the following exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(L) \otimes H^0(L)^* \xrightarrow{\alpha} H^0(L \otimes E_L) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow \cdots \tag{2.4.2}$$

where the morphism α is the same morphism as in (2.4.1). Hence $H^0(M_L \otimes E_L) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}$, i.e. E_L is simple. □

In the case of regular surfaces, under mild assumptions, which hold for example if X is a K3 surface, they are also rigid.

DEFINITION. *A vector bundle E on a smooth projective variety X is rigid if* $\mathrm{Ext}^1(E, E) = 0$.

PROPOSITION 2.14. *Let X be a smooth projective regular surface and L as above ; if the multiplication map $H^0(K_X) \otimes H^0(L) \to H^0(K_X \otimes L)$ is surjective, E_L is rigid.*

Proof. The morphism α in sequence (2.4.2) is surjective because X is regular. Let us show that $H^1(E_L) \cong 0$: indeed, by tensoring (2.2.1) with K_X in cohomology we get an exact sequence

$$0 \longrightarrow H^0(M_L \otimes K_X) \longrightarrow H^0(L) \otimes H^0(K_X) \xrightarrow{\varphi} H^0(L \otimes K_X) \longrightarrow$$

$$\longrightarrow H^1(M_L \otimes K_X) \longrightarrow H^0(L) \otimes H^1(K_X) = 0$$

Since we assumed φ surjective, we have $H^1(E_L) \cong H^1(M_L \otimes K_X) \cong 0$ by the duality theorem. Then from the exact sequence (2.4.1) it follows that $\mathrm{Ext}^1(E_L, E_L) \cong H^1(M_L \otimes E_L) \cong 0$, i.e. E_L is rigid. □

Chapitre 3

Some results on vector bundles on curves

Résumé. Soit L un fibré en droites engendré par ses sections globales sur une courbe projective lisse C de genre $g \geq 2$ sur un corps k algébriquement clos. Le fibré L définit $\phi_L : C \to \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$ et $\phi_L^* T_{\mathbb{P}^r}$ est l'image inverse sur la courbe C du fibré tangent de $\mathbb{P}^r$. En précisant un théorème dû à Paranjape, on montre que si $\deg L \geq 2g - c(C)$ alors $\phi_L^* T_{\mathbb{P}^r}$ est semistable, en disant quand il est aussi stable. De plus, on montre l'existence sur plusieurs courbes d'un fibré en droites L de degré $2g - c(C) - 1$ tel que $\phi_L^* T_{\mathbb{P}^r}$ ne soit pas semistable. Enfin, on caractérise complètement la stabilité de $\phi_L^* T_{\mathbb{P}^r}$ si C est hyperelliptique.

3.1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k and let L be a line bundle on C generated by its global sections. Let M_L be the vector bundle defined by the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \xrightarrow{e_L} L \longrightarrow 0 \tag{3.1.1}$$

where e_L is the evaluation map. We denote by E_L the dual bundle of M_L : it has degree $\deg L$ and rank $h^0(C, L) - 1$.

We recall the definition of the Clifford index of a curve.

Definition. *The Clifford index of a line bundle L on C is $c(L) = \deg L - 2(h^0(C, L) - 1)$.*

The Clifford index of a divisor D on C is the Clifford index of the associated line bundle $\mathcal{O}_C(D)$, i.e. $c(D) = c(\mathcal{O}_C(D)) = \deg D - 2 \dim |D|$.

The Clifford index of the curve C is $c(C) = \min\{c(L) / h^0(C, L) \geq 2,\ h^1(C, L) \geq 2\}$.

Clifford's theorem states that $c(C) \geq 0$, with equality if and only if C is hyperelliptic ; moreover, for any divisor D on C, $c(D) = c(K_C - D)$.

Remark 3.1. *By the Riemann-Roch theorem, $c(L) = 2g - \deg L - 2h^1(C, L)$ for any line bundle L.*

In [**28**], by using the properties of this invariant, Paranjape proves the following

Proposition 3.2. *Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections. If $c(C) \geq c(L)$ then E_L is semistable. If $h^1(C, L) = 1$ and $c(C) > 0$ or $c(C) > c(L)$, E_L is also stable.*

By completing his proof we show the following

Theorem 3.3. *Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections such that $\deg L \geq 2g - c(C)$. Then :*

1. *E_L is semistable ;*
2. *E_L is stable except when $\deg L = 2g$ and either C is hyperelliptic or $L \cong K_C(p + q)$ with $p, q \in C$.*

If C is a smooth projective $d-$gonal curve of genus $g \geq 2$ with Clifford index $c(C) = d - 2 < \frac{g-2}{2}$, we then prove the existence of a line bundle L of degree $2g - c(C) - 1$ such that E_L is not semistable.

Moreover, a theorem by Schneider (see [**31**]) states that, on a general smooth curve, E_L is always semistable : our proof also shows that one cannot replace semistable by stable in this statement.

Finally, we completely characterize the (semi)stability of E_L when C is hyperelliptic.

3.2. Proof of Theorem 3.3

We first need a lemma, shown by Paranjape in [**28**].

LEMMA 3.4. *Let F be a vector bundle on C generated by its global sections and such that $H^0(C,F^*)=0$; then $\deg F \geq \operatorname{rk} F + g - h^1(C,\det F)$ and equality holds if and only if $F = E_L$, where $L = \det F$. Moreover, if $h^1(C,\det F)\geq 2$, $\deg F \geq 2\operatorname{rk} F + c(C)$ and if equality holds $F = E_L$.*

The canonical bundle K_C is generated by its global sections and there is an exact sequence

$$0 \longrightarrow K_C^* \longrightarrow H^0(C,K_C)^* \otimes \mathcal{O}_C \longrightarrow E_{K_C} \longrightarrow 0$$

thus in cohomology we have

$$0 \longrightarrow H^0(K_C^*) \longrightarrow H^0(K_C)^* \otimes H^0(\mathcal{O}_C) \longrightarrow H^0(E_{K_C}) \longrightarrow H^1(K_C^*) \xrightarrow{\varphi} H^0(K_C)^* \otimes H^1(\mathcal{O}_C) \longrightarrow \cdots \tag{3.2.1}$$

The map φ is the dual map of $m : H^0(K_C)\otimes H^0(K_C) \to H^0(K_C^2)$, so it is injective by Noether's theorem (see [**1**], Chap.III) ; moreover, $H^0(C,K_C^*)=0$. As a consequence $H^0(C,E_{K_C}) \cong H^0(C,K_C)^* = H^1(C,\mathcal{O}_C)$ and $h^0(C,E_{K_C}) = g$.

Now we have all the tools necessary to prove Theorem 3.3.

Proof of Theorem 3.3. By Remark 3.1, if $\deg L \geq 2g - c(C)$ a fortiori $c(C) \geq c(L)$. By definition, $\deg E_L = c(L) + 2\operatorname{rk} E_L$ and $h^0(C,L) = \operatorname{rk} E_L + 1$, hence it follows by the Riemann-Roch theorem that $\deg E_L = \operatorname{rk} E_L + g - h^1(C,L)$.

Let F be a quotient bundle of E_L ; then F satisfies the hypotheses of Lemma 3.4, because it is spanned by its global sections since E_L is and $H^0(C,F^*) \subset H^0(C,E_L^*) = 0$.

Therefore, if $h^1(C,\det F) \geq 2$ we have $\deg F \geq 2\operatorname{rk} F + c(C)$; then

$$\mu(F) - \mu(E_L) \geq \frac{c(C)}{\operatorname{rk} F} - \frac{c(L)}{\operatorname{rk} E_L} = \frac{\operatorname{rk} E_L \cdot c(C) - \operatorname{rk} F \cdot c(L)}{\operatorname{rk} F \cdot \operatorname{rk} E_L} = \frac{(\operatorname{rk} E_L - \operatorname{rk} F)\cdot c(C) + \operatorname{rk} F \cdot (c(C) - c(L))}{\operatorname{rk} F \cdot \operatorname{rk} E_L} \geq 0$$

since $\operatorname{rk} E_L > \operatorname{rk} F > 0$ and $c(C) \geq c(L)$. Moreover, the inequality is strict if $c(C) > 0$ or if C is hyperelliptic and $\deg L \geq 2g+1$, because L is non-special and $c(L) < 0$.

If $h^1(C,\det F) < 2$ we still have $\deg F \geq \operatorname{rk} F + g - h^1(C,\det F)$, hence

$$\mu(F) - \mu(E_L) \geq \frac{g - h^1(\det F)}{\operatorname{rk} F} - \frac{g - h^1(L)}{\operatorname{rk} E_L} = \frac{\left[g - h^1(\det F)\right]\cdot(\operatorname{rk} E_L - \operatorname{rk} F) + \operatorname{rk} F \cdot \left[h^1(L) - h^1(\det F)\right]}{\operatorname{rk} F \cdot \operatorname{rk} E_L} > 0$$

provided that $h^1(C,L) \geq h^1(C,\det F)$, since $g - h^1(C,\det F) > 0$ follows from the hypothesis that $h^1(C,\det F) < 2$ and $g \geq 2$.

The only case remaining is $0 = h^1(C,L) < h^1(C,\det F) = 1$. We have $\deg F = \deg(\det F) \leq 2g-2$, otherwise we should have $h^1(C,\det F) = 0$; then, a fortiori, we have $\operatorname{rk} F \leq g-1$. It then follows from the previous inequalities that

$$\mu(F) - \mu(E_L) \geq \frac{(g-1)(\operatorname{rk} E_L - \operatorname{rk} F) - \operatorname{rk} F}{\operatorname{rk} F \cdot \operatorname{rk} E_L} \geq \frac{(g-1)\cdot(\operatorname{rk} E_L - \operatorname{rk} F - 1)}{\operatorname{rk} F \cdot \operatorname{rk} E_L} \geq 0 \tag{3.2.2}$$

Thus we have shown that we always have $\mu(F) - \mu(E_L) \geq 0$, i.e. E_L is semistable. In order to gain the stability of E_L, we still need to prove that $\mu(F) - \mu(E_L) > 0$ when $0 = h^1(C,L) < h^1(C,\det F) = 1$.

Suppose that $\mu(E_L) = \mu(F)$; by (3.2.2), we then have $(g-1)\cdot\operatorname{rk} E_L - g\cdot\operatorname{rk} F = 0$. Since $g \geq 2$, it follows that $(g-1)|\operatorname{rk} F \leq g-1$, i.e. $\operatorname{rk} F = g-1$, and $\operatorname{rk} E_L = g$; hence $\deg E_L = g + \operatorname{rk} E_L = 2g$ and $\mu(E_L) = 2$. Therefore, if $\deg L \neq 2g$ we cannot have $\mu(E_L) = \mu(F)$ and E_L is stable.

If $\deg L = 2g$ then E_L is stable provided that $c(C) > 0$ and $L \ncong K_C(p+q)$ with $p,q \in C$.

Indeed, since $\deg F = \operatorname{rk} F \cdot \mu(F) = 2g-2$ and $h^1(C,\det F) = 1$, we have $\det F \cong K_C$. As a consequence we have $\operatorname{rk} F + g - h^1(C,\det F) = 2g-2 = \deg F$, so $F = E_{K_C}$ by Lemma 3.4. On the other hand, F is a quotient of E_L, so there is an exact sequence

$$0 \longrightarrow W \longrightarrow E_L \longrightarrow F \longrightarrow 0 \tag{3.2.3}$$

where W is a subbundle of E_L of degree 2 and rank 1. The associated exact sequence of cohomology then is

$$0 \longrightarrow H^0(C,W) \longrightarrow H^0(C,E_L) \xrightarrow{\varphi} H^0(C,E_{K_C}) \longrightarrow H^1(C,W) \longrightarrow \cdots$$

From the exact sequence of cohomology associated to the dual sequence of (3.1.1) we see that $h^0(C,E_L) \geq g+1$ and $h^0(C,E_{K_C}) = g$ since $c(C) > 0$; hence φ cannot be injective, i.e. $H^0(C,W) \neq 0$. Thus $W \cong \mathcal{O}_C(p+q)$ with $p,q \in C$. Furthermore, it follows from (3.2.3) that

$$L = \det E_L = \det W \otimes \det F = W \otimes K_C = K_C(p+q),$$

which concludes the proof of Theorem 3.3 since this is not possible under our hypothesis. $\square$

3.3. Some line bundles of degree $2g - c(C) - 1$ with non semistable E_L

Theorem 3.3 is the best possible result that one can obtain if looking for properties of all curves.

Proposition 3.5. *Let C be a smooth projective d-gonal curve of genus $g \geq 2$ such that the Clifford index is $c(C) = d - 2 < \frac{g-2}{2}$; there is a line bundle L of degree* $\deg L = 2g - c(C) - 1$ *on C generated by its global sections and non-special such that E_L is not semistable.*

Proof. By hypothesis, $\mathfrak{g}^1_d$ computes the Clifford index. We put $N = \mathcal{O}_C(K_C - \mathfrak{g}^1_d)$: it is a line bundle of degree $2g - c(C) - 4$ and by the Riemann-Roch theorem $h^0(N) = g - c(C) - 1$. Moreover N is spanned by its global sections : assume that there is $q \in C$ such that $h^0(N(-q)) = h^0(N)$, or equivalently $h^1(N(-q)) = h^1(N) + 1$; then, by Serre's duality, we have $h^0(\mathfrak{g}^1_d + q) = h^0(\mathfrak{g}^1_d) + 1 = 3$, i.e. $\mathfrak{g}^1_d + q = \mathfrak{g}^2_{d+1}$, and this is not possible because we would have $c(\mathfrak{g}^2_{d+1}) = d - 3 < c(C)$.

Let E be an effective divisor of degree 3 on C ; we can choose E in such a way that $L = N \otimes \mathcal{O}_C(E)$ is a line bundle of degree $\deg L = 2g - c(C) - 1$, non-special and spanned by its global sections. Indeed, we have $h^1(L) = 0$ because $h^1(L) = h^0(\mathfrak{g}^1_d - E) = 0$ for a general effective divisor E ; moreover L is generated by its global sections if and only if $h^1(L(-p)) = h^1(L) = 0$ for any $p \in C$ and if E is a general effective divisor of degree 3 we have $h^1(L(-p)) = h^0(\mathfrak{g}^1_d - E + p) = 0$.

Since we have supposed that E is effective, $H^0(L \otimes N^*) \neq 0$, so we have an inclusion $N \hookrightarrow L$. Hence M_N is a subbundle of M_L, or equivalently E_N is a quotient bundle of E_L. Since $\operatorname{rk} E_L = g - c(C) - 1$ and $\operatorname{rk} E_N = h^0(N) - 1 = g - c(C) - 2$, we have

$$\mu(E_N) = 2 + \frac{c(C)}{g - c(C) - 2} < \mu(E_L) = 2 + \frac{c(C)+1}{g - c(C) - 1} \tag{3.3.1}$$

whenever $c(C) < \frac{g-2}{2}$. It then follows that E_L is not semistable. $\square$

Remark 3.6. *If C is a curve of genus $g \geq 2$ with Clifford index c, in most cases C is $(c+2)-$gonal : see* [**15**] *for further details.*

Remark 3.7. *The hypothesis that $c(C) < \frac{g-2}{2}$ leaves out only the case $c(C) = \left[\frac{g-1}{2}\right]$, i.e. the general one ; however, in* [**31**] *Schneider shows the following*

Proposition 3.8. *Let C be a general smooth curve of genus $g \geq 3$. If L is a line bundle on C generated by its global sections, E_L is semistable.*

It is worth underlining that one cannot replace semistable by stable : if C is a general curve of even genus $g = 2n$ we know that

$$c(C) = \left[\frac{g-1}{2}\right] = n - 1 = \frac{g-2}{2}, \tag{3.3.2}$$

so the proof of Proposition 3.5 shows that E_L is not stable, since one obtains $\mu(E_N) = \mu(E_L)$.

3.4. The case of hyperelliptic curves

In the case of hyperelliptic curves we completely characterize the stability of E_L.

PROPOSITION 3.9. *Let C be a smooth projective hyperelliptic curve of genus $g \geq 2$, let L be a line bundle on C generated by its global sections and such that $h^0(C, L) \geq 3$ and let H be $\mathcal{O}_C(\mathfrak{g}^1_2)$. Then :*

(1) *E_L is stable if and only if $\deg L \geq 2g + 1$;*

(2) *E_L is semistable if and only if $\deg L \geq 2g$ or there is an integer $k > 0$ such that $L = H^{\otimes k}$.*

Proof. By Theorem 3.3, if $\deg L \geq 2g$, E_L is semistable and if $\deg L \geq 2g + 1$, E_L is stable.

On the other hand E_L is not stable if $\deg L = 2g$, in which case $\mu(E_L) = 2$. Indeed, we show that H is a quotient bundle of E_L of same slope. We know that there is a surjection $E_L \twoheadrightarrow H$ if and only if there is an inclusion $H^* \rightarrowtail M_L$, if and only if $H^0(C, M_L \otimes H) \neq 0$. From the exact sequence (3.1.1) we get an exact sequence

$$0 \longrightarrow H^0(C, M_L \otimes H) \longrightarrow H^0(C, L) \otimes H^0(C, H) \longrightarrow H^0(C, L \otimes H) \longrightarrow \cdots \tag{3.4.1}$$

We then have $\dim H^0(C, L) \otimes H^0(C, H) = 2g + 2 > g + 3 = h^0(C, L \otimes H)$, so $H^0(C, M_L \otimes H) \neq 0$.

If $0 < \deg L \leq 2g - 1$ we always have $c(L) \geq 0$. If $c(L) = 0$, E_L is semistable, as it follows from the proof of Theorem 3.3 : if F is a quotient bundle of E_L, the inequality $\mu(F) - \mu(E_L) \geq 0$ still holds in each case.

Using again the exact sequence (3.4.1), since $h^0(C, L) \geq 3$, we have

$$\dim H^0(C, L) \otimes H^0(C, H) = 2h^0(C, L) > h^0(C, L) + 2 \geq h^0(C, L \otimes H).$$

Therefore, $H^0(C, M_L \otimes H) \neq 0$ and there is a surjection $E_L \twoheadrightarrow H$; furthermore,

$$\mu(E_L) = 2 + \frac{c(L)}{h^0(C, L) - 1}$$

and $\mu(H) = 2$. Thus if $c(L) > 0$ then $\mu(E_L) > \mu(H)$ and E_L is not semistable ; else, if $c(L) = 0$, $\mu(E_L) = \mu(H)$ and E_L is not stable.

The proposition then follows by Clifford's theorem : since C is hyperelliptic and $\deg L > 0$, $c(L) = 0$ if and only if there is an integer $k > 0$ such that $L = H^{\otimes k}$. $\square$

Chapitre 4

Some results on surfaces

Résumé. Soit L un fibré en droites engendré par ses sections globales sur une surface complexe projective lisse X. Le fibré L définit $\phi_L\colon X \to \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$. On étude la $L-$stabilité de $\phi_L^* T_{\mathbb{P}^r}$ quand X est une surface regulière avec $p_g = 0$, une surface K3 ou une surface abélienne. En particulier, on montre que $\phi_L^* T_{\mathbb{P}^r}$ est $L-$stable quand X est K3 et L est ample et quand X est abélienne et $L^2 \geq 14$.

4.1. Introduction

The aim of this Chapter is to study $H-$stability of vector bundles E_L in the case of smooth complex projective surfaces : since the $H-$stability of a vector bundle E is by definition the $H-$stability of its restriction $E_{|C}$ on a curve $C \in |H|$, it is natural to try to use the results known on curves to study the same problem on surfaces.

In Section 4.2 we obtain some results about regular surfaces, including the following

Theorem 4.1. *Let X be a smooth projective K3 surface over $\mathbb{C}$ and let L be an ample line bundle generated by its global sections on X ; then the vector bundle E_L is $L-$stable.*

Then in Section 4.3 we study the case of abelian surfaces, showing the following

Theorem 4.2. *Let X be a smooth projective abelian surface over $\mathbb{C}$ and let L be a line bundle on X generated by its global sections such that $L^2 \geq 14$. Then the vector bundle E_L is $L-$stable.*

Throughout this Chapter we will work over the field of complex numbers.

4.2. About regular surfaces

Before restricting to the case of regular surfaces, let us see a few statements which hold for every surface.

Lemma 4.3. *Let F be a vector bundle of rank 2 generated by its global sections on a smooth projective surface X and assume moreover that $h^0(\det F) = 2$. Then there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow \det F \longrightarrow 0 \tag{4.2.1}$$

Proof. We cannot have $F = \mathcal{O}_X^2$ because $h^0(\det F) = 2$; then, since F is of rank 2 generated by its global sections, we have $h^0(F) \geq 3$. There is a section $s \in H^0(X, F)$ which is zero only in a finite number of points and we have the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow \mathcal{I}_Z \det F \longrightarrow 0 \tag{4.2.2}$$

where Z is the zero locus of s. In cohomology we obtain

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, F) \longrightarrow H^0(X, \mathcal{I}_Z \det F) \longrightarrow \cdots$$

Since $h^0(F) \geq 3$, we get $h^0(\mathcal{I}_Z \det F) \geq 2$, but $h^0(\mathcal{I}_Z \det F) \leq h^0(\det F) = 2$. Since $\det F$ is generated by its global sections, from $h^0(\mathcal{I}_Z \det F) = h^0(\det F) = 2$ it follows that $\mathcal{I}_Z \det F = \det F$ and $Z = \varnothing$. Therefore the sequence (4.2.2) becomes (4.2.1). $\square$

PROPOSITION 4.4. *Let X be a smooth projective surface over $\mathbb{C}$ and let L be a line bundle on X generated by its global sections. Let C be a smooth irreducible curve on X such that $H^1(L(-C)) = 0$. Then $(E_L)_{|C} = E_{(L_{|C})} \oplus \mathcal{O}_C^r$, with $r = h^0(L(-C))$.*

Proof. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with L, in cohomology we get

$$0 \longrightarrow H^0(X, L(-C)) \longrightarrow H^0(X, L) \longrightarrow H^0(X, L_{|C}) \longrightarrow 0$$

So we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L^*_{|C} \arrow[r] \arrow[d] & H^0(X, L_{|C})^* \otimes \mathcal{O}_C \arrow[r] \arrow[d] & E_{(L_{|C})} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^*_{|C} \arrow[r] \arrow[d] & H^0(X, L)^* \otimes \mathcal{O}_C \arrow[r, "e_L"] \arrow[d] & (E_L)_{|C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}_C^r \arrow[r] \arrow[d] & \mathcal{O}_C^r \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.2.3}$$

By the snake lemma, the third column is exact. Since the middle column trivially splits we have a retraction $r : \mathcal{O}_C^r \longrightarrow H^0(X, L)^* \otimes \mathcal{O}_C$ and this induces a retraction $e_L \circ r : \mathcal{O}_C^r \longrightarrow (E_L)_{|C}$. Hence also the third column splits and $(E_L)_{|C} = E_{(L_{|C})} \oplus \mathcal{O}_C^r$. □

COROLLARY 4.5. *Let X be a smooth projective regular surface over $\mathbb{C}$ such that $p_g = 0$ and let C be a smooth irreducible curve on X of genus $g \geq 2$ such that $\mathcal{O}_X(C)$ and $L = \mathcal{O}_X(K_X + C)$ are generated by their global sections ; then E_L is $C-$semistable and it is also $C-$stable if $c(C) > 0$.*

Proof. By Proposition 4.4 $(E_L)_{|C} \cong E_{(L_{|C})}$, since $r = p_g = 0$; on the other hand, $L_{|C} = K_C$, so the statement follows from Theorem 2.5. □

When $r \neq 0$, the restriction to the curve is no longer semistable, but in the case of K3 surfaces Proposition 4.4 is enough to prove $C-$stability.

Proof of Theorem 4.1 Let $C \in |L|$ be a smooth irreducible curve of genus $g \geq 2$. By Proposition 4.4 we have $(E_L)_{|C} = E_{K_C} \oplus \mathcal{O}_C$, since $L_{|C} \cong K_C$; moreover $\mu(E_L) = \frac{2g-2}{g} < 2$. Let us suppose that $g \geq 3$: if $g = 2$ then C is hyperelliptic and we will deal with the case $c(C) = 0$ later. Let F be a torsion-free quotient sheaf of E_L of rank $0 < \operatorname{rk} F < g$; then $F_{|C}$ is a quotient of $(E_L)_{|C}$ and we can suppose that it is a vector bundle on C. There is a diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d] & (E_L)_{|C} \arrow[r] \arrow[d] & E_{K_C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d] & F_{|C} \arrow[r] \arrow[d] & G \oplus \tau \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where G is a vector bundle generated by its global sections, W is either $\mathcal{O}_C$ or 0 and τ is a torsion sheaf on C, hence $\deg W = 0$ and length $\tau \geq 0$. So we get $\mu(F) = \frac{\deg G + \text{length}\,\tau}{\operatorname{rk} F}$.

(1) If $\operatorname{rk} G = 0$, $\operatorname{rk}(F) = 1$ and we always have $\mu(F) \geq 2$. Indeed, we cannot have $F = \mathcal{O}_X$, since this would imply the existence of a nontrivial section $\mathcal{O}_X \hookrightarrow M_L$, in contradiction with $H^0(X, M_L) = 0$. Hence $F = \mathcal{O}_X(D)$ with $D > 0$ an effective base-point free divisor such that $D.C \geq 1$ because C is ample ; since $\mathcal{O}_C(D)$ is globally generated we have then $D.C \geq 2$.

(2) If $\operatorname{rk} G > 0$, G is generated by its global sections such that $H^0(C, G^*) = 0$, because G^* is a subbundle of M_{K_C} and $H^0(C, M_{K_C}) = 0$; the hypotheses of Lemma 3.4 then hold and, since $\mu(F) \geq \frac{\deg G}{\operatorname{rk} F}$, we have :

(a) if $h^1(\det G) < 2$, since $g \geq 3$, by Lemma 3.4
$$\mu(F) \geq 1 + \frac{g-2}{\operatorname{rk} G + 1} > 1 + \frac{g-2}{g} = \mu(E_L).$$

(b) If $h^1(\det G) \geq 2$, by Lemma 3.4
$$\mu(F) \geq 2 + \frac{c(\det G) + \deg \tau - 2}{\operatorname{rk} G + 1} \geq 2 > \mu(E_L)$$
if $c(\det G) \geq 2$, in particular if $c(C) \geq 2$, but also if $c(\det G) = 1$ and $\deg \tau > 0$.

This shows that $\mu(F) > \mu(E_L)$ in the case $c(C) \geq 2$.

We now deal with the case $c(C) = 1$. We can repeat the above proof by applying Lemma 3.4 and it does not work only if $h^1(\det G) \geq 2$, $\deg \tau = 0$ and $c(\det G) = 1$. If $g = 3$ then $\mu(E_L) = \frac{4}{3}$ and we always have $\mu(F) > \frac{4}{3}$.

From now on we assume $g \geq 4$; then either the curve is trigonal or a smooth plane quintic of genus $g = 6$ (see [**24**]).

(1) If there is a $\mathfrak{g}^1_3$ on C, the only line bundles which compute the Clifford index are $\mathcal{O}_C(\mathfrak{g}^1_3)$ and $\mathcal{O}_C(K_C - \mathfrak{g}^1_3)$.

(a) If $\det G = \mathcal{O}_C(\mathfrak{g}^1_3)$, since $h^1(\det G) \geq 2$, by Lemma 3.4 we have $\deg G \geq 2\operatorname{rk} G + 1$, hence in this case $\operatorname{rk} G = 1$. Then $\operatorname{rk} F = 2$ and $\det F_{|C} = \mathcal{O}_C(\mathfrak{g}^1_3)$; it follows that $\det F = \mathcal{O}_X(D)$ with $D.C = 3$. By the Hodge index theorem then, since $g \geq 4$, we have $D^2 \leq \frac{9}{2g-2} < 2$, so $D^2 = 0$ and $D = kE$ with $k \geq 1$ and E an elliptic curve ; since $D.C = 3$ and $C.E \geq 2$, this implies $k = 1$ and $h^0(\mathcal{O}_X(D)) = 2$; by Lemma 4.3, it follows from $h^1(\det F^*) = 0 = \operatorname{Ext}^1(\mathcal{O}_X, \det F)$ that $F = \mathcal{O}_X \oplus \det F$, hence $h^0(F^*) > 0$, which is impossible.

(b) If $\det G = \mathcal{O}_C(K_C - \mathfrak{g}^1_3)$ we have $\deg G = 2g - 5$ and $\operatorname{rk} G \leq g - 3$ by Lemma 3.4, hence
$$\mu(F) \geq \frac{2g-5}{\operatorname{rk} G + 1} \geq \frac{2g-5}{g-2} = 2 - \frac{1}{g-2} > \mu(E_L)$$
if $g > 4$. If $g = 4$ we have $\deg G = 3$ and we fall in the former case.

(2) If there is a $\mathfrak{g}^2_5$ on C, the genus is $g = 6$ and the only line bundle which computes the Clifford index is $\mathcal{O}_C(\mathfrak{g}^2_5) \cong \mathcal{O}_C(K_C - \mathfrak{g}^2_5)$.

If $\det G = \mathcal{O}_C(\mathfrak{g}^2_5)$, since $h^1(\det G) \geq 2$, by Lemma 3.4 $\deg G \geq 2\operatorname{rk} G + 1$, hence $\operatorname{rk} G \leq 2$ and $\operatorname{rk} F \leq 3$. Therefore we get
$$\mu(F) = \frac{5}{\operatorname{rk} G + 1} \geq \frac{5}{3} = \mu(E_L)$$
Let us investigate whether equality can hold or not ; suppose that $\operatorname{rk} F = 3$. Since F is of rank > 2 generated by its global sections, there is a short exact sequence
$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow V \longrightarrow 0 \tag{4.2.4}$$
with V of rank 2 generated by its global sections such that $\det V = \det F = \mathcal{O}_X(D)$ with $D.C = 5$. By the Hodge index theorem then $D^2 \leq 2$; however the case $D^2 = 2$ cannot occur,

since otherwise $(C-2D)^2 = -2$ and by Riemann-Roch theorem at least one between $C-2D$ and $2D-C$ would be effective, contradicting $(C-2D).C = 0$ and the ampleness of C. If $D^2 = 0$, then $D = kE$ with $k \geq 1$ and E an elliptic curve; since $D.C = 5$ and $C.E \geq 2$, this implies $k = 1$ and $h^0(\mathcal{O}_X(D)) = 2$, so by Lemma 4.3 there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} V \longrightarrow \det V \longrightarrow 0$$

and in cohomology we obtain $h^1(V^*) = h^1(V) = 0$. As a consequence we have $\mathrm{Ext}^1(\mathcal{O}_X, V) = 0$ and $F = \mathcal{O}_X \oplus V$, impossible since it would imply $h^0(F^*) > 0$.

Then $\mu(F) > \mu(E_L)$ also if $c(C) = 1$.

Suppose now that C is a hyperelliptic curve; then the morphism $\phi_L : X \longrightarrow \mathbb{P}^g$ induces a double covering $\pi : X \longrightarrow F$ where $F \subset \mathbb{P}^g$ is a rational surface of degree $g-1$ which is either smooth or a cone over a rational normal curve (see [**3**], page 129). Let $i : F \hookrightarrow \mathbb{P}^g$ be the embedding and $H = i^*\mathcal{O}_{\mathbb{P}^g}(1)$ the ample hyperplane section of F such that $\pi^*H = L$; whenever E_H is $H-$stable, this yields the $L-$stability of E_L, because π is a finite covering (see [**22**], Lemma 3.2.2).

If $g = 2$ then $F = \mathbb{P}^2$ (see [**3**], page 129) and it is well-known that its tangent bundle is $\mathcal{O}_{\mathbb{P}^2}(1)-$stable (see Lemma 2.1 and [**22**] Section 1.4).

If $g \geq 3$, we have $H^2 = g-1$. On the surface F we have the short exact sequence

$$0 \longrightarrow H^* \longrightarrow H^0(F,H)^* \otimes \mathcal{O}_F \longrightarrow E_H \longrightarrow 0 \tag{4.2.5}$$

We know that the curve H is rational, so $p_a(H) = 0$; we consider a smooth curve $\Gamma \in |2H|$. By the adjunction formula we have $0 = p_a(H) = 1 + \frac{1}{2}(H^2 + H.K_F)$, so we get $H.K_F = -H^2 - 2 = -g-1$; using the adjunction formula once more we then obtain

$$p_a(\Gamma) = 1 + \frac{1}{2}(\Gamma^2 + \Gamma.K_F) = 1 + 2H^2 + H.K_F = g-2$$

Since $g \geq 3$ we have $p_a(\Gamma) \geq 1$. Since H is ample, we deduce $H^0(F, \mathcal{O}_F(-H)) = H^1(F, \mathcal{O}_F(-H)) = 0$ (see [**25**]). Then from the short exact sequence

$$0 \longrightarrow \mathcal{O}_F(H-\Gamma) \longrightarrow \mathcal{O}_F(H) \longrightarrow \mathcal{O}_\Gamma(H) \longrightarrow 0$$

and from the associated cohomology sequence it follows that $H^0(F, \mathcal{O}_F(H)) \cong H^0(F, \mathcal{O}_\Gamma(H))$, hence $(E_H)_{|\Gamma} = E_{\mathcal{O}_\Gamma(H)}$.

Moreover, $\deg \mathcal{O}_\Gamma(H) = H.\Gamma = 2g-2 > 2p_a(\Gamma) = 2g-4$. Since $\mathcal{O}_\Gamma(H)$ is a line bundle on a smooth projective curve Γ of genus ≥ 1 of degree $> 2p_a(\Gamma)$, $(E_H)_{|\Gamma}$ is stable (see [**14**]).

Since E_H is $2H-$stable, it is also $H-$stable and this ends the proof. □

REMARK 4.6. *Throughout the proof the ampleness of L is needed only when C is a smooth plane quintic of genus $g = 6$ to show that we cannot have equality between slopes. Indeed, if we only assume that L is generated by its global sections and $L^2 \geq 2$, E_L is still $L-$semistable and also $L-$stable unless C is a smooth plane quintic of genus $g = 6$.*

4.3. About abelian surfaces

In this section we study the same problem when X is an abelian surface over $\mathbb{C}$ and we give the proof of Theorem 4.2.

PROPOSITION 4.7. *Let X be an abelian surface over $\mathbb{C}$; then there is no irreducible hyperelliptic curve of genus $g \geq 6$ and no irreducible trigonal curve of genus $g \geq 8$ on X.*

Proof. Take $d = 2$ or 3 and suppose that there is a $d-$gonal irreducible curve C of genus $g \geq 2d+2$ on X. Then there is an exact sequence of sheaves on X

$$0 \longrightarrow F^* \longrightarrow H^0(g^1_d) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_C(g^1_d) \longrightarrow 0$$

where F is a vector bundle of rank 2 such that $c_1(F) = C$ and $c_2(F) = d$. Dualizing the above exact sequence we get

$$0 \longrightarrow \mathcal{O}_X^2 \longrightarrow F \longrightarrow \mathcal{O}_C(K_C - g^1_d) \longrightarrow 0$$

It follows from the assumption on the genus that $c_1(F)^2 - 4c_2(F) = 2g-2-4d > 0$, so F is Bogomolov unstable (see [**30**]). Therefore, there is a line bundle $\mathcal{O}_X(A)$ on X such that $\mu(\mathcal{O}_X(A)) > \mu(F)$, i.e. $2A.C > C^2$, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow F \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}_X(B) \longrightarrow 0$$

with $A + B = C$, $A.B + \deg \mathcal{I}_Z = d$ and $(A - B)^2 > 0$ (see [**30**]). Hence we can construct the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_X^2 \arrow[d, "i"] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{O}_X(A) \arrow[r] & F \arrow[r] \arrow[d] & \mathcal{I}_Z \otimes \mathcal{O}_X(B) \arrow[r] & 0 \\
 & & \mathcal{O}_C(K_C - g^1_d) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Since i is an isomorphism outside C, $h^0(\mathcal{I}_Z \otimes \mathcal{O}_X(B)) > 0$ and B is effective. By the Hodge index theorem $A^2B^2 \leq (A.B)^2 \leq d^2$. Since $K_X = 0$, A^2 and B^2 are even numbers and $A^2 > B^2$ because $2A.C > C^2$, hence we must have $B^2 \leq 2$.

If $B^2 = 2$, then $d = 3$ and $A^2 = 4$ and we would have $6 - 2A.B > 0$, so $A.B \leq 2$ in contradiction with $A^2B^2 = 8$. Therefore $B^2 = 0$, which means that $B = kE$ where E is an elliptic curve and $k \geq 1$; on the other hand we know that $0 \leq A.B \leq d$. In fact $A.B > 0$, otherwise by the Hodge index theorem it would follow $B = 0$ against the fact that $h^0(\mathcal{I}_Z \otimes \mathcal{O}_X(B)) > 0$; hence $1 \leq kA.E \leq d$. Since $A.E = 1$ would imply that A itself is elliptic, the only possibility is $k = 1$ and $A.B > 1$. In this case we have $h^0(B) = 1$, hence by the snake lemma we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "s"] & \mathcal{O}_X^2 \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(A) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & \mathcal{I}_Z \otimes \mathcal{O}_X(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau \arrow[r] \arrow[d] & \mathcal{O}_C(K_C - g^1_d) \arrow[r] \arrow[d] & \tau' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where τ and τ' are two torsion sheaves with support respectively on the zero-locus of s and σ. Hence the exactness of the third line implies that C is reducible, against our assumptions. $\square$

Proof of Theorem 4.2. Since L is generated by its global sections such that $L^2 \geq 14$, the general member of $|L|$ is a smooth irreducible curve of genus $g \geq 8$. Given a nontrivial $\alpha \in \mathrm{Pic}^0(X)$, we can find $C \in |L \otimes \alpha^{-1}|$ smooth irreducible of genus $g \geq 8$. The $L-$stability of E_L is equivalent to the $C-$stability of E_L. Since we have $H^0(\alpha) = H^1(\alpha) = 0$, it follows from Proposition 4.4 that $(E_L)_{|C} \cong E_{(L_{|C})}$. Moreover, $L_{|C} \cong K_C \otimes \alpha_{|C}$, so by Theorem 3.3 E_L is $C-$stable if $c(C) \geq 2$. By the hypothesis on the genus of C and by Proposition 4.7 the cases $c(C) = 0, 1$ cannot occur, so there is nothing more to prove. □

REMARK 4.8. *In the case $g(C) \leq 7$ the same proof shows the $L-$stability of E_L if $c(C) \geq 2$. Moreover, it is possible to show that E_L is $L-$stable also if either C is a smooth plane quintic of genus $g = 6$ or if C is a trigonal curve of genus $g = 4$.*

4.4. Further developments

The techniques employed in this chapter could work also on surfaces with higher geometric genus and irregularity. Indeed we checked that the same proof works, assuming that the Clifford index of the curve considered satisfies a lower bound depending on the invariants of the surface, with the only exception of the case in which the restriction of the quotient F of E_L satisfies the following short exact sequence

$$0 \longrightarrow \mathcal{O}_C^r \longrightarrow F_{|C} \longrightarrow \tau \longrightarrow 0$$

Unfortunately this case would cause the slope inequality to fail and we are not able to understand whether such a case can occur or not.

Another possible generalization would be to extend these results to higher dimensions but of course this would require many more vanishing cohomology groups and as a consequence the statements obtained would be less interesting.

Chapitre 5

Involutions of irreducible symplectic fourfolds

Résumé. On étude les involutions des variétés irréductibles holomorphes symplectiques, en particulier les involutions symplectiques. Après avoir rappellé la définition et les propriétés des variétés irréductibles holomorphes symplectiques et l'outil principal qu'on utilise, la formule de Lefschetz holomorphe introduite par Atiyah-Singer dans [**2**], on remarque que les composantes irréductibles du lieu fixe d'une involution symplectique i sont des sous-variétés symplectiques lisses et donc si X est une variété irréductible holomorphe de dimension 4 les composantes du lieu fixe de i sont soit des points fixes isolés soit des surface lisses K3 ou abéliennes. On démontre qu'une involution symplectique i a toujours au moins 12 points fixes isolés et une surface fixée. On conjecture qu'une involution symplectique ne fixe jamais une surface abélienne, et dans ce cas i aurait 28 points fixes isolés et une surface K3 fixée. Enfin on donne des arguments en faveur de la conjecture, en montrant que dans les exemples connus les involutions symplectiques la vérifient, en regardant les involutions naturelles sur le schéma d'Hilbert d'une surface K3, les involutions de la variété de Fano d'une cubique lisse de $\mathbb{P}^5$ induites par une involution de $\mathbb{P}^5$ et les involutions de la variété recouvrement double d'une sextique EPW induites par une involution de $\mathbb{P}^5$.

5.1. Introduction

In this chapter we are going to study involutions of irreducible symplectic fourfolds and their fixed points. In particular we are going to concentrate on symplectic involutions, i.e. those which preserve the symplectic form.

The study of symplectic involutions and more generally of automorphisms of finite order on K3 surfaces has been started by Nikulin in [**26**]. Since irreducible holomorphic symplectic manifolds are the natural generalization of K3 surfaces in higher dimension, Beauville started to study the same problems for such manifolds in [**4**]. Many authors have studied the problem from different view-points, here we want to mention only the papers by Boissière [**9**] and Boissière-Sarti [**10**] on natural involutions and the paper by Beauville [**7**] in which he deals with the case of antisymplectic involutions.

5.2. Irreducible holomorphic symplectic manifolds

Let us recall first of all the definition of irreducible holomorphic symplectic manifold; for all the details on this subject the reader may refer to [**5**] and to Part III of [**20**].

Definition. *A compact Kähler manifold X is irreducible holomorphic symplectic if it is simply connected and admits a symplectic 2-form $\omega \in H^{2,0}(X)$ everywhere non degenerate and unique up to multiplication by a nonzero scalar.*

It follows immediately from the definition that we have $H^0(\Omega_X) \cong H^1(\mathcal{O}_X) = 0$, since X is simply connected; moreover the existence of a symplectic 2-form implies that the complex dimension of X is always even and that K_X is trivial. From the definition it follows that the Hodge structure of the second cohomology ring $H^2(X,\mathbb{C})$ is $H^2(X,\mathbb{C}) \cong \mathbb{C}\omega \oplus H^{1,1}(X) \oplus \mathbb{C}\bar{\omega}$ and we have an isomorphism between TX and Ω^1_X.

Not many examples of irreducible symplectic manifolds are known. Here we briefly describe those in dimension 4 that we need in the next sections.

The Hilbert scheme of a K3 surface. Let S be a smooth K3 surface and let $X = S^{[2]}$ be the Hilbert scheme of S of $0-$schemes of length 2; then X can be constructed in the following way :

$$\begin{tikzcd} \mathrm{Bl}_\Delta(S\times S) \arrow[r] \arrow[d] & X \arrow[d] \\ S\times S \arrow[r] & S^{(2)} \end{tikzcd}$$

as the blow-up along the diagonal Δ of the symmetric product of S. For further details see [**5**]. All the other families we are going to consider turn out to be deformation equivalent to this one and hence will have the same cohomology.

In particular let us recall that $b_2(X) = 23$ and that the Hodge diamond is the following

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 21 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 21 & & 232 & & 21 & & 1 \end{array}$$

The Fano variety of a smooth cubic in $\mathbb{P}^5$. Let X be a smooth cubic hypersurface in $\mathbb{P}^5$ and F the Fano variety of X, i.e. the variety of lines contained in X. In their paper of 1985 [**8**] Beauville and Donagi show that this is an irreducible holomorphic symplectic fourfold deformation equivalent to the former family. Moreover, they show that there is an isomorphism of Hodge structures $\alpha : H^4(X,\mathbb{Z}) \cong H^2(F,\mathbb{Z})$.

The double cover of an EPW sextic. This example has been introduced and intensively studied by O'Grady in [**27**] and many other papers. Starting from a 6-dimensional vector space V and from a general enough Lagrangian subspace $A \subset \wedge^3 V$, the subvariety $Y_A := \left\{v \in \mathbb{P}(V)/(v \wedge \wedge^2 V) \cap A \neq 0\right\}$ turns out to be a hypersurface of degree 6 of the type described by Eisenbud-Popescu-Walter; Y_A is not smooth, but it has a smooth double cover X_A that is an irreducible holomorphic symplectic fourfold when A is general enough. We will discuss this more in detail in Section 5.8.

5.3. Holomorphic Lefschetz Theorem

Let us briefly recall the Holomorphic Lefschetz theorem by Atiyah-Singer (see [**2**]), following the paper by Donovan [**13**], where the reader can find all the details and the proofs that we are skipping. In order to keep the notation as simple as possible we limit our presentation to the case of involutions.

Let $Z \subset \mathrm{Fix}(i)$ be an irreducible component of the fixed point set of an involution i on a smooth projective variety X. Let N_Z^* be the dual of the normal bundle of Z; since TZ is fixed by di and on the other hand i is non degenerate, from the exact sequence

$$0 \longrightarrow TZ \longrightarrow TX_{|Z} \longrightarrow N_Z \longrightarrow 0$$

it follows that N_Z is the eigensheaf corresponding to -1.

Let us consider a vector bundle F on X and let $\eta : i^*F \longrightarrow F$ be a morphism such that the composite morphism $\eta \circ i^*\eta$ is the identity; then there is an induced action i^* on the vector space of global sections $\Gamma(F)$. Hence the involution i and η induce an action on the cohomology ring $H^*(X,F)$ of the vector bundle F that we will always denote by i^* for the sake of simplicity.

Moreover η induces an involution $\eta_{|Z} : F_{|Z} \longrightarrow F_{|Z}$ with eigenvalues ± 1. Throughout all that follows we will denote $F^+_{|Z}$ and $F^-_{|Z}$ the eigensheaves respectively fixed by $\eta_{|Z}$ and associated to -1.

THEOREM 5.1. ***Holomorphic Lefschetz-Riemann-Roch formula*** *Let X be a smooth projective variety of dimension d, i an involution of X and F a vector bundle on X; let $\eta : i^*F \longrightarrow F$ be a morphism such that $\eta \circ i^*\eta = \mathrm{id}_F$ and let i^* be the induced action on $H^*(X,F)$. Let $Z \subset \mathrm{Fix}(i)$ be* *an irreducible component of the fixed point set. Then the following formula holds :*

$$\sum_{j=0}^{d}(-1)^j \mathrm{Tr}(i^*_{|H^j(F)}) = \sum_{\substack{Z\subset \mathrm{Fix}(i)\\ irreducible}} \int_Z \frac{\mathrm{Td}(Z).\left[\mathrm{ch}(F^+_{|Z}) - \mathrm{ch}(F^-_{|Z})\right]}{\sum_{p\geq 0}\mathrm{ch}(\wedge^p N_Z^*)}$$

5.4. Fixed loci

The property of preserving the symplectic form induces limitations on the irreducible components of the locus of fixed points. Let us remark some important properties.

LEMMA 5.2. *Let X be a projective smooth variety and $f : X \longrightarrow X$ a periodic endomorphism ; then each component of the fixed point set* $\mathrm{Fix}(f)$ *is smooth.*

Proof. See [**13**], Lemma 4.1. □

In the case we are interested in we have more than smoothness.

PROPOSITION 5.3. *Let X be an irreducible holomorphic symplectic manifold and i a symplectic involution on X. Then the irreducible components of* $\mathrm{Fix}(i)$ *are symplectic subvarieties of X.*

Proof. Let $Z \subset \mathrm{Fix}(i)$ be an irreducible component of the fixed point set of dimension $d > 0$. We need to prove that the restriction to Z of the symplectic form ω gives a symplectic form on Z. We know that $TX_{|Z} \cong TZ \oplus N_Z$ and that TZ and N_Z are respectively the eigensheaves associated to ± 1. Given $z \in Z$, since i is symplectic, T_zZ and $N_{Z,z}$ are orthogonal and hence both symplectic. □

REMARK 5.4. *In particular if X has dimension 4, the irreducible components can be either isolated fixed points or K3 and abelian surfaces.*

5.5. Symplectic involutions

Now we are ready to show the main result. We will show that there are few different possibilities for the nature of the fixed locus of a symplectic involution on an irreducible holomorphic symplectic fourfold such that $b_2 = 23$. By [**21**], when X is an irreducible holomorphic symplectic fourfold such that $b_2 = 23$, the canonical map $S^2H^2(X,\mathbb{C}) \to H^4(X,\mathbb{C})$ is an isomorphism and, as we already said in Section 5.2, this is the case for the family of Hilbert schemes $S^{[2]}$ of a K3 surface S and for their deformations.

THEOREM 5.5. *Let X be an irreducible holomorphic symplectic fourfold such that $b_2(X) = 23$ and let i be a symplectic involution of X. Let τ be the trace of i^* on $H^{1,1}(X)$, N and K respectively the numbers of isolated fixed points and of K3 surfaces of fixed points. Then only the following cases can occur :*

(1) $\tau = -3$, $N = 12$ *and* $K = 0$ *;*

(2) $\tau = 3$, $N = 36$ *and* $K = 0$ *;*

(3) $\tau = 5$, $N = 28$ *and* $K = 1$.

Moreover in the first two cases i fixes at least one abelian surface.

In fact, we conjecture that only the last case can occur.

CONJECTURE. *Let X and i be as in Theorem 5.5 ; the fixed locus of i cannot contain an abelian surface.*

In the next sections we will provide evidence for this conjecture verifying it in some of the known examples of irreducible symplectic fourfolds such that $b_2(X) = 23$.

Proof of Theorem 5.5. Let us apply the holomorphic Lefschetz Riemann-Roch formula discussed in Section 5.3 to the cohomology of the vector bundles $\mathcal{O}_X$, Ω^1_X and Ω^2_X.

The vector bundle $\mathcal{O}_{\mathbf{X}}$. We know that $h^{0,0} = h^{2,0} = h^{4,0} = 1$ and $h^{1,0} = h^{3,0} = 0$; on the other hand, we know that $H^{2,0}(X) = \langle \omega_X \rangle$ and $H^{4,0}(X) = \langle \omega_X^2 \rangle$, hence they are fixed by the involution and the Lefschetz number is $L(i) = \Sigma(-1)^i \mathrm{Tr}(i^*_{|H^{i,0}(X)}) = 3$.

For each fixed surface Y of $\mathrm{Fix}\,(i)$ we have to calculate

$$\int_Y \mathrm{Td}(Y).(1 + \mathrm{ch}(N_Y^*) + \mathrm{ch}(\det N_Y^*))^{-1}$$

since the only eigenvalue of di is -1 and the rank of N_Y^* is 2. From the short exact sequence

$$0 \longrightarrow TY \longrightarrow TX_{|Y} \longrightarrow N_Y \longrightarrow 0$$

we get $c_1(N_Y^*) = -c_1(N_Y) = -c_1(TX_{|Y}) + c_1(TY) = 0$ and

$$c_2(N_Y^*) = c_2(N_Y) = c_2(TX_{|Y}) - c_2(TY) - c_1(N_Y)c_1(TY) = c_2(X).\,[Y] - c_2(Y).$$

The Lefschetz formula for $\mathcal{O}_X$ becomes then

$$3 = \sum_{i(p)=p} \frac{1}{\det(1 - \mathrm{d}i_{|T_p})} + \sum_{i(S_j)=S_j} \int_{S_j} \frac{1 + \frac{1}{12}c_2(S_j)}{4 + c_2(S_j) - c_2(X).\,[S_j]} \tag{5.5.1}$$

The restriction of di to T_p is $-\mathrm{id}_{\mathbb{C}^4}$, hence $\det(1 - \mathrm{d}i_{|T_p}) = 2^4 = 16$.

An easy computation gives

$$\frac{1 + \frac{1}{12}c_2(S_j)}{4 + c_2(S_j) - c_2(X).\,[S_j]} = \frac{1}{4} - \frac{1}{24}c_2(S_j) + \frac{1}{16}c_2(X).\,[S_j]$$

Let us write $a_j := \int_{S_j} c_2(X).\,[S_j]$; since $\int_{S_j} c_2(S_j) = 24$ if S_j is K3 and $\int_{S_j} c_2(S_j) = 0$ if S_j is abelian, from (5.5.1) it follows that

$$3 = \frac{N}{16} - K + \frac{1}{16} \sum_{i(S_j)=S_j} a_j \tag{5.5.2}$$

The vector bundle $\mathbf{\Omega^1_X}$. We know that $h^{0,1} = h^{2,1} = h^{4,1} = 0$ (see [**21**] Theorem 1) and $h^{1,1} = h^{3,1} = 21$, since $H^{1,1}(X) \cong H^{3,1}(X)$; moreover this isomorphism is compatible with i^* since it is given by product with ω_X and i is symplectic. The Lefschetz number is $L(i, \Omega^1_X) = -2\tau$.

In this case we get

$$L(i, \Omega^1_X) = \sum_{i(Y)=Y} \int_Y \frac{\mathrm{Td}(Y).(\mathrm{ch}(\Omega^+_{|Y}) - \mathrm{ch}(\Omega^-_{|Y}))}{(1 + \mathrm{ch}(N_Y^*) + \mathrm{ch}(\det N_Y^*))}$$

where $\Omega^+_{|Y}$ and $\Omega^-_{|Y}$ are respectively the subbundle of $\Omega^1_{X|Y}$ fixed by the action of the dual of d$i_{|Y}$ and the subbundle on which the dual of d$i_{|Y}$ has eigenvalue -1.

When Y is an isolated fixed point $p \in X$ we get

$$\frac{-4}{\det(1 - \mathrm{d}i_{|T_p})} = -\frac{1}{4}$$

When Y is a fixed surface, we have to calculate $(\mathrm{ch}(\Omega^+_{|Y}) - \mathrm{ch}(\Omega^-_{|Y}))$. Since $\Omega^+_{|Y} = TY$ and $\Omega^-_{|Y} = N_Y^*$, we have

$$(\mathrm{ch}(\Omega^+_{|Y}) - \mathrm{ch}(\Omega^-_{|Y})) = 2 - c_2(Y) - 2 + c_2(X).\,[Y] - c_2(Y) = c_2(X).\,[Y] - 2c_2(Y)$$

From

$$\left(\frac{1}{4} - \frac{1}{24}c_2(Y) + \frac{1}{16}c_2(X).\,[Y]\right).\,(c_2(X).\,[Y] - 2c_2(Y)) = \frac{1}{4}c_2(X).\,[Y] - \frac{1}{2}c_2(Y)$$

it follows then that the Lefschetz formula for Ω^1_X is

$$-2\tau = -\frac{N}{4} - 12K + \frac{1}{4} \sum_{i(S_j)=S_j} a_j \tag{5.5.3}$$

The vector bundle $\Omega^2_{\mathbf{X}}$. We know that $h^{0,2} = h^{4,2} = 1$, $h^{2,2} = 232$ and $h^{1,2} = h^{3,2} = 0$: indeed, from $S^2H^2(X,\mathbb{C}) \cong H^4(X,\mathbb{C})$ (see [**21**]) it follows

$$H^{2,2}(X) \cong H^{2,0}(X) \otimes H^{0,2}(X) \oplus S^2H^{1,1}(X) \cong \mathbb{C} \oplus S^2H^{1,1}(X)$$

Let us write $\sigma := \mathrm{Tr}(i^*_{|H^{2,2}(X)})$; we need to deduce σ from τ.

If i^* is of type (a,b) on $H^{1,1}(X)$, we have $\tau = a - b$ and $h^{1,1} = a + b$; on the other hand $H^{1,1}(X) = H^{1,1}_+ \oplus H^{1,1}_-$ implies that

$$H^{2,2}(X) \cong \mathbb{C} \oplus S^2H^{1,1}_+ \oplus S^2H^{1,1}_- \oplus H^{1,1}_+ \otimes H^{1,1}_-$$

Hence $\sigma = 1 + \frac{a(a+1)}{2} + \frac{b(b+1)}{2} - ab = 1 + \frac{h^{1,1}}{2} + \frac{\tau^2}{2}$.

The Lefschetz number is then $L(i,\Omega^2_X) = 2 + \sigma = 3 + \frac{21}{2} + \frac{\tau^2}{2} = \frac{27+\tau^2}{2}$.

We need to know which are the subbundles of $\Omega^2_{X|Y}$ associated to the eigenvalues 1 and -1. We have $(\Omega^2_{X|Y})^+ \cong \wedge^2\Omega^+_{|Y} \oplus \wedge^2\Omega^-_{|Y} \cong \det TY \oplus \det N^*_Y$ and $(\Omega^2_{X|Y})^- \cong TY \otimes N^*_Y$.

When Y is an isolated fixed point $p \in X$ we get

$$\frac{6}{\det(1 - \mathrm{d}i_{|T_p})} = \frac{3}{8}$$

When Y is a fixed surface, we get instead

$$\begin{aligned}(\mathrm{ch}((\Omega^2_{X|Y})^+) - \mathrm{ch}((\Omega^2_{X|Y})^-)) = \mathrm{ch}(\mathcal{O}^2_Y) - \mathrm{ch}(TY)\mathrm{ch}(N^*_Y) = 2 - 4 + 2c_2(X).\,[Y] = \\ = 2c_2(X).\,[Y] - 2\end{aligned}$$

Since

$$\left(\frac{1}{4} - \frac{1}{24}c_2(Y) + \frac{1}{16}c_2(X).\,[Y]\right).\,(2c_2(X).\,[Y] - 2) = -\frac{1}{2} + \frac{3}{8}c_2(X).\,[Y] + \frac{1}{12}c_2(Y)$$

the Lefschetz formula for Ω^2_X becomes

$$\frac{27+\tau^2}{2} = \frac{3N}{8} + 2K + \frac{3}{8}\sum_{i(S_j)=S_j} a_j \tag{5.5.4}$$

The system. We have thus obtained the following linear system

$$\begin{cases} 3 = \frac{N}{16} - K + \frac{1}{16}\sum_{i(S_j)=S_j} a_j \\ -2\tau = -\frac{N}{4} - 12K + \frac{1}{4}\sum_{i(S_j)=S_j} a_j \\ \frac{27+\tau^2}{2} = \frac{3N}{8} + 2K + \frac{3}{8}\sum_{i(S_j)=S_j} a_j \end{cases} \tag{5.5.5}$$

from which, by eliminating $\sum_{i(S_j)=S_j} a_j$, we deduce

$$\begin{cases} -\tau^2 + 4\tau + 33 = N \\ \tau^2 - 9 = 16K \end{cases} \tag{5.5.6}$$

On the other hand it must be $K \geq 0$ and $N \geq 0$, hence τ must satisfy the following :

$$\begin{cases} \tau \leq -3 \text{ or } \tau \geq 3 \\ 2 - \sqrt{37} \leq \tau \leq 2 + \sqrt{37} \end{cases}$$

Moreover the second equation of (5.5.6) implies that τ is odd and that it cannot be 7, since otherwise K would not be an integer.

If we replace the solutions we found in (5.5.5), we get

(1) $\sum_{i(S_j)=S_j} a_j = 36$ when $\tau = -3$;

(2) $\sum_{i(S_j)=S_j} a_j = 12$ when $\tau = 3$;

(3) $\sum_{i(S_j)=S_j} a_j = 36$ when $\tau = 5$.

Hence if there is a symplectic involution satisfying the first or the second line of the table it must have a fixed abelian surface. This ends the proof. □

COROLLARY 5.6. *Let X and i be as in Theorem 5.5 ; then :*

(1) *i has always at least 12 isolated fixed points and 1 fixed surface ;*

(2) *i fixes at most 1 K3 surface and in this case it has 28 isolated fixed points.*

5.6. The Hilbert scheme of a K3 surface

As a first evidence to our conjecture, we will show that the natural symplectic involution on the Hilbert scheme of a K3 surface fixes exactly 28 isolated points and 1 K3 surface.

Let S be a smooth K3 surface and let X be the Hilbert scheme of S of $0-$schemes of length 2 (see Section 5.2 for some details on the construction) ; given an involution σ of S, there is an involution $i = \sigma^{[2]}$ induced by it : such an involution is said to be natural. For further details on natural involutions the reader is referred to [**10**] and [**9**].

Here we want to remark only that if σ is symplectic then also i will preserve the symplectic form on X. Moreover, Nikulin showed in [**26**] that a symplectic involution on a smooth K3 surface fixes 8 isolated points. Hence, on X the isolated fixed points will be all the couples $\{p, q\}$ where $p, q \in \mathrm{Fix}(\sigma)$ are distinct ; this gives $\binom{8}{2} = 28$ isolated fixed points. The fixed K3 surface is the closure in X of the surface made of the points $\{p, \sigma(p)\}$ with $p \in S \setminus \mathrm{Fix}(\sigma)$.

Let us study the deformations of the couple (X, i). We will show that there are nontrivial deformations, i.e. deformations that cannot be obtained from a deformation of (S, σ).

The infinitesimal deformations of X are unobstructed and there is a canonical isomorphism $j : H^2(S, \mathbb{C}) \oplus \mathbb{C}e \longrightarrow H^2(X, \mathbb{C})$ (see [**5**]), where e is the class of the exceptional divisor.

PROPOSITION 5.7. *Let S be a smooth K3 surface and σ a symplectic involution on S ; let $X = S^{[2]}$ be the Hilbert scheme of S and $i = \sigma^{[2]}$ the natural symplectic involution on X. Then the infinitesimal deformations of the couple (X, i) are parametrized by $H^{1,1}(X)^i = j(H^{1,1}(S)^\sigma) \oplus \mathbb{C}e$.*

Proof. We have the following diagram

$$\begin{tikzcd} \mathrm{Def}(S,\sigma) \arrow[r] \arrow[d] & \mathrm{Def}(S) \arrow[d] \\ \mathrm{Def}(X,i) \arrow[r] & \mathrm{Def}(X) \end{tikzcd}$$

Looking at the tangent spaces at 0 we thus get

$$\begin{tikzcd} H^{1,1}(S)^\sigma \arrow[rr] \arrow[d] & & H^{1,1}(S) \arrow[d] \\ H^{1,1}(X)^i \arrow[r] & H^{1,1}(X) = j(H^{1,1}(S)) \oplus \mathbb{C}e & \end{tikzcd}$$

Since $\tau = 5$ and $h^{1,1}(S) = 20$, we have $\dim H^{1,1}(X)^i = 13$; on the other hand $\dim H^{1,1}(S)^\sigma = 12$ by Theorem 5.3 and all natural automorphisms leave globally invariant the exceptional divisor (see [**10**]), hence $i^*e = e$ and $e \in H^{1,1}(X)^i$. As a consequence we see that $H^{1,1}(X)^i = j(H^{1,1}(S)^\sigma) \oplus \mathbb{C}e$. □

5.7. The Fano variety of a smooth cubic

Let X be a smooth cubic in $\mathbb{P}^5$ and let F be the variety of lines of X ; it is an irreducible holomorphic symplectic fourfold (see [**8**]). We want to investigate which involutions σ of $\mathbb{P}^5$ induce involutions of X and hence of F and of which kind these ones are.

We have the following situation :

σ	X	$p \in X$ s.t. $\sigma(p) = p$
$[X_0, .., X_5]$ $\downarrow$ $[-X_0, X_1, .., X_5]$	$X_0^2 L + G$ with $L \in \mathbb{C}[X_1, .., X_5]_1$ $G \in \mathbb{C}[X_1, .., X_5]_3$	$[1, 0, .., 0]$, $[0, y_1, .., y_5] \in V(G)$
$[X_0, .., X_5]$ $\downarrow$ $[-X_0, -X_1, X_2, .., X_5]$	$X_0^2 L_0 + X_1^2 L_1 + X_0 X_1 L_2 + G$ with $L_i \in \mathbb{C}[X_2, .., X_5]_1$ $G \in \mathbb{C}[X_2, .., X_5]_3$	$[x_1, x_2, 0, .., 0]\ \forall [x_1, x_2] \in \mathbb{P}^1$, $[0, 0, y_1, .., y_4] \in V(G)$
$[X_0, .., X_5]$ $\downarrow$ $[-X_0, -X_1, -X_2, X_3, X_4, X_5]$	$X_0^2 L_0 + \cdots + X_2^2 L_5 + G$ with $L_i \in \mathbb{C}[X_3, .., X_5]_1$ $G \in \mathbb{C}[X_3, .., X_5]_3$	$[x_1, x_2, x_3, 0, 0, 0]$ $\forall [x_1, x_2, x_3] \in \mathbb{P}^2$, $[0, 0, 0, y_1, .., y_3] \in V(G)$

In [**8**] the authors show that $\alpha : H^4(X, \mathbb{Z}) \cong H^2(F, \mathbb{Z})$ is an isomorphism of Hodge structures ; via this isomorphism we have $\alpha(H^{2,0}(F)) = H^{3,1}(X)$. By Griffiths' theorem on the cohomology of hypersurfaces in $\mathbb{P}^n$ (see [**32**] §18 Théorème 18.1), $H^{3,1}(X)$ is generated by the residue of a meromorphic 5-form of $\mathbb{P}^5$ with poles of order 2 along X, i.e.

$$\Omega = \sum (-1)^i X_i \frac{dX_0 \wedge .. \wedge d\hat{X}_i \wedge .. \wedge dX_5}{P^2}$$

where P is a polynomial defining X. Hence σ induces on F a symplectic involution i if and only if $\sigma^* \Omega = \Omega$ and this is true only in the second case.

Let us study in more detail the locus of fixed points on F in the 3 cases.

(1) In the first case, the lines of fixed points are the lines contained in the cubic threefold $G = X_0 = 0$ in $\mathbb{P}^4$; they are parametrized by the Fano variety of this cubic.

All other lines fixed by the involution pass through 2 fixed points, hence they can be parametrized as

$$\lambda [1, 0, 0, .., 0] + \mu [0, a_1, .., a_5]$$

Replacing in the equation of X we get

$$G(a_1, .., a_5) = L(a_1, .., a_5) = 0$$

which gives a cubic surface in $\mathbb{P}^3$.

(2) In the symplectic case, we claim that the fixed locus is given by 28 isolated points and one K3 surface.

Indeed there are the lines of fixed points, i.e. $X_2 = ... = X_5 = 0$ and the 27 lines on the cubic $G = X_0 = X_1 = 0$, which give the 28 points. All other lines fixed by the involution pass through 2 fixed points, hence they can be parametrized as

$$\lambda [a_1, a_2, 0, .., 0] + \mu [0, 0, b_1, .., b_4]$$

Replacing in the equation of X we get

$$a_1^2 L_0(b_1, .., b_4) + a_2^2 L_1(b_1, .., b_4) + a_1 a_2 L_2(b_1, .., b_4) = 0$$

equation which defines a divisor of bidegree $(2, 1)$ in $\mathbb{P}^1 \times V(G) \subset \mathbb{P}^1 \times \mathbb{P}^3$, which is a K3 surface.

(3) In the third case, the lines of fixed points are the lines in the $\mathbb{P}^2$ defined by $X_3 = X_4 = X_5 = 0$. All other lines fixed by the involution pass through 2 fixed points, hence they can be parametrized as

$$\lambda [a_1, a_2, a_3, 0, 0, 0] + \mu [0, 0, 0, b_1, b_2, b_3]$$

Replacing in the equation of X we get an equation of bidegree $(2, 1)$ in $\mathbb{P}^2 \times E$, where E is the elliptic curve in $\mathbb{P}^2$ given by $G = X_0 = X_1 = X_2 = 0$; hence we obtain a surface S which is a conic bundle over E.

When i is symplectic, let us study $\mathrm{Def}(F, i)$ and compare it with $\mathrm{Def}(X, \sigma)$.

PROPOSITION 5.8. *Let X be a smooth cubic in $\mathbb{P}^5$ and let F be the variety of lines of X ; let σ be the involution of $\mathbb{P}^5$ such that $\sigma^*\Omega = \Omega$ and i the symplectic involution induced by σ on F. Then the infinitesimal deformations of the couple (F, i) are parametrized by $H^{1,1}(F)^i \cong H^1(X, TX)^\sigma$.*

Proof. We have (see [**32**] Corollary 18.12 and Lemma 18.15) $H^1(X, TX) \cong R^3_P \cong H^{2,2}(X)_0$, where R^3_P is the degree 3 component of the Jacobian ring of P, but on the other hand (see [**8**]) we know that $H^{1,1}(F)_0 \cong H^{2,2}(X)_0$. Hence also the invariant parts will be isomorphic, i.e. all deformations of (F, i) are obtained by deforming (X, σ) and taking the Fano variety of the deformation with the induced involution. □

5.8. O'Grady's example

As a last example let us see what happens in the case of the double cover of an EPW sextic. Let V be a 6-dimensional vector space, $\mathbb{P}(V) \cong \mathbb{P}^5$; on $\wedge^3 V$ the wedge product $\wedge : \wedge^3 V \times \wedge^3 V \longrightarrow \wedge^6 V$ induces a sympletic form ω by choosing an isomorphism $\wedge^6 V \cong \mathbb{C}$. Let us take a Lagrangian subspace $A \subset \wedge^3 V$ and let us define $Y_A := \left\{v \in \mathbb{P}(V)/(v \wedge \wedge^2 V) \cap A \neq 0\right\}$; for general A, Y_A is a hypersurface of degree 6 of the type described by Eisenbud-Popescu-Walter. Such a hypersurface is not smooth, but for a general A it has a smooth double cover X_A which is an irreducible holomorphic symplectic fourfold (see [**27**]). We will show that the involution of V such that $\dim V^+ = 4$ induces, when A is general enough, a symplectic involution on X_A which fixes exactly 28 isolated points and one K3 surface.

Let F be the vector bundle given on fibers by $F_v = v \wedge \wedge^2 V$ for all $v \in \mathbb{P}^5$; there is an isomorphism $F \cong \Omega^3_{\mathbb{P}^5}(3)$. We look at the morphism $\lambda_A : F \longrightarrow \frac{\wedge^3 V}{A} \otimes \mathcal{O}_{\mathbb{P}^5}$ given on fibers by

$$v \wedge \alpha \in F_v \mapsto [v \wedge \alpha] \in \frac{\wedge^3 V}{A}$$

for $v \in \mathbb{P}^5$. Since the two sheaves considered have both rank 10, $\operatorname{Coker} \lambda_A$ is a torsion sheaf on $\mathbb{P}^5$.

Let us define $\xi_A = \zeta_A \otimes \mathcal{O}_{Y_A}(-3)$ where ζ_A is a coherent sheaf on Y_A such that $j_*\zeta_A = \operatorname{Coker} \lambda_A$; there is an isomorphism $\alpha_A : \xi_A \longrightarrow \xi_A^*$ that gives $\mathcal{O}_{Y_A} \oplus \xi_A$ the structure of a commutative $\mathcal{O}_{Y_A}-$algebra. We define $X_A = \operatorname{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$; the structure map $f : X_A \longrightarrow Y_A$ is finite of degree 2. The fourfold X_A is smooth whenever

$$A \in \mathbb{LG}^0(\wedge^3 V) = \{A \in \mathbb{LG}(\wedge^3 V)/\mathbb{P}(A) \cap G(3,6) = \emptyset \text{ and } \dim(v \wedge \wedge^2 V) \cap A \leq 2 \text{ for all } v \in V\}$$

The double cover is ramified over $W_A = \{v \in Y_A / \dim(v \wedge \wedge^2 V) \cap A = 2\}$.

LEMMA 5.9. *Let $i : V \longrightarrow V$ be an involution ; if $i(A) = A$, i induces an involution $\hat{\imath}$ on X_A.*

Proof. We clearly have $i^*\Omega^3_{\mathbb{P}^5}(3) \cong \Omega^3_{\mathbb{P}^5}(3)$.

$$\begin{tikzcd}
F_{i(v)} \arrow[r] \arrow[d, "i"] & \frac{\wedge^3 V}{i(A)} \arrow[d, "i"] \\
F_v \arrow[r, "\lambda_{A,v}"] & \frac{\wedge^3 V}{A}
\end{tikzcd}$$

and hence also

$$\begin{tikzcd}
i^*F \arrow[r] \arrow[d] & \frac{\wedge^3 V}{i(A)} \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[d] \\
F \arrow[r, "\lambda_A"] & \frac{\wedge^3 V}{A} \otimes \mathcal{O}_{\mathbb{P}^5}
\end{tikzcd} \tag{5.8.1}$$

If $i(A) = A$ then $i(Y_A) = Y_A$. Indeed, $Y_A = \left\{v \in \mathbb{P}(V)/(v \wedge \wedge^2 V) \cap A \neq 0\right\}$ is invariant for i as soon as A is.

In order to prove that i induces an involution on X_A we also need to show that $i^*\zeta_A \cong \zeta_A$ and that the morphism $\alpha_A : \zeta_A \longrightarrow \zeta_A^*$ commutes with i. It follows from diagram (5.8.1) that $i^* \operatorname{Coker} \lambda_A \cong \operatorname{Coker} \lambda_A$ and this implies $i^*\zeta_A \cong \zeta_A$.

Moreover we know that $\alpha_A = \beta_A \otimes \mathrm{id}_{\mathcal{O}_{Y_A}(-3)}$ (see [**27**] Proof of Proposition 4.4) where β_A satisfies the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\lambda_A"] \arrow[d] & A^* \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r] \arrow[d] & j_*\zeta_A \arrow[r] \arrow[d, "\beta_A"] & 0 \\
0 \arrow[r] & A \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r, "\lambda_A^*"] & F^* \arrow[r] & \mathrm{Ext}^1(j_*\zeta_A, \mathcal{O}_{\mathbb{P}^5}) \arrow[r] & 0
\end{tikzcd}$$

Everything in this diagram is invariant for i, hence β_A and consequently α_A commutes with i. This shows that X_A is fixed by an involution $\hat{\imath}$ such that

$$\begin{tikzcd}
X_A \arrow[r, "\hat{\imath}"] \arrow[d, "f"] & X_A \arrow[d, "f"] \\
Y_A \arrow[r, "i"] & Y_A
\end{tikzcd} \tag{5.8.2}$$

□

Remark 5.10. *Given i involution on Y_A, there are two involutions i_1 and i_2 on X_A which fit into diagram (5.8.2) : they can be obtained one from each other by composition with the covering involution i_A, i.e. the involution which exchanges the sheets of f. Since the covering involution i_A is antisymplectic, i.e. $i_A^*\omega_{X_A} = -\omega_{X_A}$, we deduce that one involution will be symplectic and the other antisymplectic. In all what follows we will denote by $\hat{\imath}$ the unique symplectic involution induced on X_A by i.*

In order to apply Theorem 5.5 to $\hat{\imath}$ we need to show that there are Lagrangian subspaces $A \in \mathbb{LG}^0(\wedge^3 V)$ invariant for i such that X_A is smooth.

Given the decomposition $V = V^+ \oplus V^-$ as direct sum of eigenspaces of i, we get

$$\wedge^3 V = (\wedge^3 V)^+ \oplus (\wedge^3 V)^- = \wedge^3 V^+ \oplus (V^+ \otimes \wedge^2 V^-) \oplus (V^- \otimes \wedge^2 V^+) \oplus \wedge^3 V^- \tag{5.8.3}$$

A subspace $A \subset \wedge^3 V$ is invariant under i if and only if it can be written $A^+ \oplus A^-$, with $A^+ \subset \wedge^3 V^+ \oplus (V^+ \otimes \wedge^2 V^-)$ and $A^- \subset (V^- \otimes \wedge^2 V^+) \oplus \wedge^3 V^-$.

We need to check that for such a general Lagrangian A we have X_A smooth, i.e. that A does not contain any decomposable tensors and that $\dim A \cap F_l \leq 2$ for all $l \in \mathbb{P}(V)$ (see [**27**]).

Proposition 5.11. *If $\dim V^+ = 5$ or 3, X_A is not smooth.*

Proof. If $\dim V^+ = 5$, X_A is not smooth. Indeed, we have either $\dim A^+ \geq 5$ or $\dim A^- \geq 5$. In the first case, since in $\mathbb{P}(\wedge^3 V^+) \cong \mathbb{P}^9$ decomposable tensors are parametrized by $G(2,5)$ of dimension 6, it must be $\mathbb{P}(A^+) \cap G(2,5) \neq \emptyset$. If $\dim A^- \geq 5$, since $\mathbb{P}(V^- \otimes \wedge^2 V^+) \cong \mathbb{P}^9$ and decomposable tensors are parametrized by $\mathbb{P}(V^-) \times G(2,5)$ of dimension 6, we get $\mathbb{P}(A^-) \cap (\mathbb{P}(V^-) \times G(2,5)) \neq \emptyset$.

If $\dim V^+ = 3$, X_A is not smooth. Indeed, if $\dim A^+ \geq 6$ we immediately find a decomposable tensor in $\mathbb{P}(A^+)$, since $\mathbb{P}(V^+) \times G(2,3)$ is a subvariety of dimension 4 of the variety of decomposable tensors in $\mathbb{P}^9$. An analogous dimensional count shows that there is a decomposable in $\mathbb{P}(A^-)$ if $\dim A^- \geq 6$.

Let us suppose that $\dim A^+ = \dim A^- = 5$. We claim that there is $v \in V^+$ such that $\dim(v \wedge \wedge^2 V) \cap A \geq 3$ and this shows that A is not in $\mathbb{LG}(\wedge^3 V)^0$.

First of all, let us remark that there is $v \in V^+$ such that $(v \wedge \wedge^2 V) \cap A^+ \neq 0$ if and only if $p(A^+) \cap (v \wedge \wedge^2 V^-) \neq 0$ where $p : \wedge^3 V^+ \oplus (V^+ \otimes \wedge^2 V^-) \longrightarrow V^+ \otimes \wedge^2 V^-$ is the projection. Indeed, either $A^+ \cap \wedge^3 V^+ \neq 0$, and in this case there is a decomposable in A^+ and the proof ends, or $\dim p(A^+) = 5$. Hence there is a finite number of vectors $v \in V^+$ such that $(v \wedge \wedge^2 V) \cap A^+ \neq 0$: indeed, we have seen that $\dim \mathbb{P}(p(A^+)) = 4$ and on the other hand $\mathbb{P}(V^+) \times \mathbb{P}(\wedge^2 V^-)$ is a 4-dimensional subvariety of $\mathbb{P}(V^+ \otimes \wedge^2 V^-) \cong \mathbb{P}^8$, so they intersect in a finite number of points.

Let $v \in V^+$ such a vector ; let us show that $\dim(v \wedge \wedge^2 V) \cap A^- \geq 2$. Since $A^- \subset (\wedge^2 V^+ \otimes V^-) \oplus \wedge^3 V^-$, this is equivalent to $\dim(v \wedge V^+ \otimes V^-) \cap A^- \geq 2$. Let us write $A' := A^- \cap (\wedge^2 V^+ \otimes V^-)$; we have $\dim A' = 4$ otherwise there would be a decomposable in A^-. Hence the morphism $\wedge v : A' \longrightarrow \wedge^3 V^+ \otimes V^-$ has nontrivial kernel W.

On the other hand we have shown that there is $\tau \in \wedge^2 V \setminus \{0\}$ such that $v \wedge \tau \in A^+$ is nonzero. The composition $\wedge\tau \circ \wedge v : A' \longrightarrow \wedge^3 V^+ \otimes V^- \longrightarrow \wedge^6 V$ must then be zero, since by hypothesis A is Lagrangian and this happens only if $A^+ \wedge A^- = 0$. It follows that $\dim W \geq 2$, since otherwise $\wedge v$ would be surjective and $\wedge\tau$ should be identically zero, which would imply $\tau \in \wedge^2 V^+$ and give a decomposable tensor in A. □

We are left with the case in which $\dim V^+ = 4$, but we need a deeper analysis to understand it. First of all let us remark that in this case i is symplectic on $\wedge^3 V$, since $\det i = 1$, and this implies also that the decomposition (5.8.3) of $\wedge^3 V$ in eigenspaces of i is orthogonal with respect to the symplectic form ω : if $\alpha \in (\wedge^3 V)^+$ and $\beta \in (\wedge^3 V)^-$, $(\det i)(\alpha \wedge \beta) = i(\alpha) \wedge i(\beta) = -\alpha \wedge \beta$, hence $\alpha \wedge \beta = 0$.

Let us also recall a standard fact from linear algebra.

REMARK 5.12. *If we have 3 vector spaces W, E_1 and E_2 such that :*

(1) $\dim W = \dim E_1 = \dim E_2$ *;*

(2) $W \subset E_1 \oplus E_2$ *;*

(3) $W \cap E_i = 0$ *for* $i = 1, 2$*,*

then there is an isomorphism $f : E_1 \longrightarrow E_2$ such that W is the graph of f.

Since the third assumption implies that the two projections $p_i : W \longrightarrow E_i$ are isomorphisms, the isomorphism $f = p_2 \circ p_1^{-1}$ makes the deal.

LEMMA 5.13. *If $\dim V^+ = 4$ and f_1, f_2 are a basis of V^-, let $A = A^+ \oplus A^-$ be a Lagrangian subspace of $\wedge^3 V$ such that $\wedge^3 V^+ \cap A^+ = 0$, $A^+ \cap (\wedge^2 V^- \otimes V^+) = 0$ and $A^- \cap (\mathbb{C} f_i \otimes \wedge^2 V^+) = 0$ for $i = 1, 2$. Then :*

(1) *there is a self-adjoint operator $u : \wedge^2 V^+ \to \wedge^2 V^+$ such that $A^- = \left\{ f_1 \wedge x + f_2 \wedge u(x) / x \in \wedge^2 V^+ \right\}$;*

(2) *$A^+ = \{ f_1 \wedge f_2 \wedge v + \phi(v) / v \in V^+ \}$ where $\phi : V^+ \to \wedge^3 V^+$ is a linear isomorphism such that $v \wedge \phi(w) = w \wedge \phi(v)$ for all $v, w \in V^+$.*

Proof. A is Lagrangian if for all $v, w \in A$ we have $v \wedge w = 0$. Here A will be Lagrangian as soon as A^+ and A^- are Lagrangian respectively in $\wedge^3 V^+ \oplus (V^+ \otimes \wedge^2 V^-)$ and in $(V^- \otimes \wedge^2 V^+)$, since $A^+ \wedge A^- = 0$ comes from the orthogonality of the decomposition (5.8.3).

(1) First of all let us remark that given u and A^- as in the statement, A^- is Lagrangian : for all $x, y \in \wedge^2 V^+$ we have

$$(f_1 \wedge x + f_2 \wedge u(x)) \wedge (f_1 \wedge y + f_2 \wedge u(y)) = f_1 \wedge f_2 \wedge (-x \wedge u(y) + u(x) \wedge y) = 0 \qquad (5.8.4)$$

Now let us consider $A^- \subset V^- \otimes \wedge^2 V^+ = (\mathbb{C} f_1 \otimes \wedge^2 V^+) \oplus (\mathbb{C} f_2 \otimes \wedge^2 V^+)$; then by Remark 5.12 there is an isomorphism $u : \wedge^2 V^+ \to \wedge^2 V^+$ such that A^- is its graph ; (5.8.4) tells us that u is self-adjoint because A^- is a Lagrangian subspace.

(2) If A^+ and ϕ satisfies the statement, A^+ is Lagrangian in $\wedge^3 V^+ \oplus (\wedge^2 V^- \otimes V^+)$: indeed, for all $v, w \in V^+$ we have

$$(f_1 \wedge f_2 \wedge v + \phi(v)) \wedge (f_1 \wedge f_2 \wedge w + \phi(w)) = f_1 \wedge f_2 \wedge (v \wedge \phi(w) + \phi(v) \wedge w) = 0 \qquad (5.8.5)$$

Viceversa, given a Lagrangian subspace A^+ in $\wedge^3 V^+ \oplus (\wedge^2 V^- \otimes V^+)$ such that $\wedge^3 V^+ \cap A^+ = 0$ and $A^+ \cap (\wedge^2 V^- \otimes V^+) = 0$, by Remark 5.12 there is an isomorphism $\phi : V^+ \to \wedge^3 V^+$ linear such that A^+ is its graph. Since A^+ is Lagrangian, from (5.8.5) we deduce that ϕ satisfies $v \wedge \phi(w) = w \wedge \phi(v)$ for all $v, w \in V^+$. □

LEMMA 5.14. *Using the notation of Lemma 5.13, if u has 6 distinct eigenvalues and no decomposable eigenvector in $\wedge^2 V^+$, then A^- does not contain any decomposable tensor and there is a basis of eigenvectors $x_1, \ldots, x_6 \in \wedge^2 V^+$ such that u is diagonalizable.*

Proof. If $v \wedge w_1 \wedge w_2 \in A^-$ we can suppose $v \in V^-$ and it follows that there must be $\lambda \in \mathbb{C}$ such that $u(w_1 \wedge w_2) = \lambda w_1 \wedge w_2$. This is against our assumption that u has no decomposable eigenvectors, hence there are no decomposable tensors in A^-. Let $x \in \wedge^2 V^+$ be an eigenvector of u ; since $Q(x) \neq 0$, we have an orthogonal decomposition $\wedge^2 V^+ = \mathbb{C}x \oplus V'$ such that V' is invariant for u. By iterating this reasoning we then get an orthogonal basis of eigenvectors. □

Lemma 5.15. *Using the notation of Lemma 5.13, A^+ does not contain any decomposable tensor.*

Proof. If $v \wedge w_1 \wedge w_2 \in A^+$ then we can suppose $v \in V^+$ and it follows that $w_1 \wedge w_2$ is a decomposable in $\wedge^2 V^+ \oplus \wedge^2 V^-$ which is only possible if $w_1 \wedge w_2 \in \wedge^2 V^+$ or $w_1 \wedge w_2 \in \wedge^2 V^-$, against the fact that $A^+ \cap \wedge^3 V^+ = 0$ and $A^+ \cap (\wedge^2 V^- \otimes V^+) = 0$. □

Let us define $\mathbb{LG}(\wedge^3 V)^*$ to be the set of all $A \in \mathbb{LG}(\wedge^3 V)$ such that A admits a decomposition as in Lemma 5.13 with u satisfying also the hypothesis of Lemma 5.14. It follows from what we said that it is an open set inside $\mathbb{LG}(\wedge^3 V)$. Indeed, up to a base change of V^-, $\wedge^3 V^+ \cap A^+ = 0$, $A^+ \cap (\wedge^2 V^- \otimes V^+) = 0$ and $A^- \cap (\mathbb{C}f_i \otimes \wedge^2 V^+) = 0$ for $i = 1, 2$ are all open conditions, since $\mathbb{P}(A^-)$ and $\mathbb{P}(\mathbb{C}f_i \otimes \wedge^2 V^+)$ are both 5-dimensional linear subspaces of $\mathbb{P}^{11}$ and $\dim \mathbb{P}(A^+) = \dim \mathbb{P}(\wedge^3 V^+) = \dim \mathbb{P}(\wedge^2 V^- \otimes V^+) = 3$ in $\mathbb{P}^7$.

Proposition 5.16. *If $A = A^+ \oplus A^- \in \mathbb{LG}(\wedge^3 V)^*$ then $A \in \mathbb{LG}(\wedge^3 V)^0$.*

Proof. Let us remark that if $v \wedge w_1 \wedge w_2 \in A$, such a decomposable is associated to a 3-dimensional vector subspace $W \subset V$ which must therefore verify $\dim W \cap V^+ \geq 1$. We can hence suppose that $v \in \mathbb{P}(V^+) \cap Y_A$ and hence v is fixed by the involution i. Moreover it must be $\dim(v \wedge \wedge^2 V) \cap A \geq 2$, since otherwise we would have found a decomposable inside A^+ or A^-, which is not possible by Lemma 5.14 and 5.15 since $A \in \mathbb{LG}(\wedge^3 V)^*$.

To conclude the proof we need to analyze better the fixed points of i on Y_A and for this purpose we need to recall here the construction of the quadric line complex : for further details and all the proofs the reader is referred to Chapter 6 of [**19**].

Lines l_x in $\mathbb{P}^3$ are parametrized by the Grassmannian $G = G(2,4) \subset \mathbb{P}^5$, which is defined by the Plücker quadric equation $x \wedge x = 0$. Given another smooth quadric F in $\mathbb{P}^5$, the intersection $X = F \cap G$ is the so-called quadric line complex. Given $p \in \mathbb{P}^3$ we want to understand which lines of our complex pass through p. Let $\sigma(p)$ be the set of all $x \in X$ such that $p \in l_x$; it is a 2-plane contained in G and we look at its intersection with F, which is a conic in $\sigma(p)$. The set S of points $p \in \mathbb{P}^3$ such that $F \cap \sigma(p)$ is a singular conic is shown to be a singular Kummer surface of degree 4, called the associated Kummer surface of X. The singular locus R of S is made of 16 ordinary double points, which are precisely the ones such that $F \cap \sigma(p)$ is a double line.

Given $x \in X$, the corresponding line l_x is singular if there is $p \in l_x$ such that $\sigma(p)$ is tangent to F in x. The set Σ of points $x \in X$ such that l_x is singular is a smooth minimal K3 surface and there is a morphism $\pi : \Sigma \longrightarrow S$ defined by $\pi(x) = p \in l_x$ such that $\sigma(p) = T_x F$. In fact, π is the blow-up of S along R. There is also a morphism $\pi' : \Sigma \longrightarrow S^*$ defined by $\pi'(x) = h \supset l_x$ such that $\sigma(h) = \{y \in G / l_y \subset h\}$ is tangent to F in x. There is a commutative diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & S \arrow[dl, "\delta"] \\ & S^* & \end{tikzcd}$$

where $\delta(v) = T_v S$ for all $v \in S$.

Another characterization of Σ is the following. Let $x \wedge Qx = 0$ and $x \wedge Q'x = 0$ be the two quadratic forms defining the two quadrics G and F, where Q and Q' are two symmetric matrices; $\Sigma = G \cap F \cap H$ where H is the quadric hypersurface corresponding to the matrix $Q'Q^{-1}Q'$. Since we will need it later, let us remark that, by standard linear algebra, whenever Q' has distinct eigenvalues it is possible to suppose that $Q = I$ and at the same time to diagonalize Q' (see Chapter XII §6 of [**17**]) : hence we can find homogeneous coordinates $[X_0, \ldots, X_5] \in \mathbb{P}^5$ such that G and F are respectively defined by the equations $\sum_{i=0}^5 X_i^2 = 0$ and $\sum_{i=0}^5 \lambda_i X_i^2 = 0$. Using such coordinates then H turns out to be defined by $\sum_{i=0}^5 \lambda_i^2 X_i^2 = 0$.

PROPOSITION 5.17. *Let i be an involution on V such that $\dim V^+ = 4$ and let $A \in \mathbb{LG}(\wedge^3 V)^*$ be invariant for i. Then the fixed locus $\mathrm{Fix}(i)$ of i on Y_A is the union of 6 isolated fixed points $q_1, \dots, q_6 \in \mathbb{P}(V^+)$, one smooth quadric Q and a singular Kummer surface S of degree 4 in $\mathbb{P}^3$.*

Proof. The fixed points of i on Y_A are precisely the intersections $Y_A \cap \mathbb{P}(V^+)$ and $Y_A \cap \mathbb{P}(V^-)$. Given $v \in V$ we want to understand when $(v \wedge \wedge^2 V) \cap A \neq 0$; since $(v \wedge \wedge^2 V) \cap A$ is fixed by the involution, this intersection splits into

$$(v \wedge \wedge^2 V) \cap A = ((v \wedge \wedge^2 V)^+ \cap A^+) \oplus ((v \wedge \wedge^2 V)^- \cap A^-)$$

where $(v \wedge \wedge^2 V)^+$ and $(v \wedge \wedge^2 V)^-$ are the intersections of $(v \wedge \wedge^2 V)$ respectively with $(\wedge^3 V)^+$ and $(\wedge^3 V)^-$. We are going to investigate each of these summands separately.

(1) If $v \in V^+$ we have $(v \wedge \wedge^2 V)^+ = v \wedge (\wedge^2 V^+ \oplus \wedge^2 V^-)$ and $(v \wedge \wedge^2 V)^- = v \wedge (V^+ \otimes V^-)$.
   - If $\alpha \in (v \wedge \wedge^2 V)^+ \cap A^+$, there is $\tau \in \wedge^2 V^+$ such that $\alpha = v \wedge \tau + v \wedge f_1 \wedge f_2$ and on the other hand since $\alpha \in A^+$ we have $\alpha = f_1 \wedge f_2 \wedge v + \phi(v)$ by Lemma 5.13. Hence we get $\phi(v) = v \wedge \tau$ and this happens if and only if $v \wedge \phi(v) = 0$: this equation defines a quadric Q in $\mathbb{P}^3 \cong \mathbb{P}(V^+)$.
   - If $\alpha \in (v \wedge \wedge^2 V)^- \cap A^-$, there are $y_1, y_2 \in V^+$ such that $\alpha = v \wedge y_1 \wedge f_1 + v \wedge y_2 \wedge f_2$ and on the other hand there is $x \in \wedge^2 V^+$ such that $\alpha = f_1 \wedge x + f_2 \wedge u(x)$ by Lemma 5.13. Comparing the two expressions we get

$$\begin{cases} x = v \wedge y_1 \\ u(x) = v \wedge y_2 \end{cases} \tag{5.8.6}$$

   and the forms $x \in \wedge^2 V^+$ solutions of (5.8.6) are exactly those who satisfy $x \wedge x = x \wedge u(x) = u(x) \wedge u(x) = 0$ in $\mathbb{P}(\wedge^2 V^+) \cong \mathbb{P}^5$. Hence $x \in \Sigma$, the smooth K3 surface associated to the quadric line complex defined above, where as F we consider the quadric hypersurface defined by $x \wedge u(x) = 0$.
   Keeping the notation we used above, we claim that $\pi(x) = v \in S$. Indeed, $\pi(x) \in l_x$ is the point such that $T_x F = \sigma(\pi(x))$; hence every line corresponding to a point of $T_x F$ passes through $\pi(x)$, which can be recovered as the intersection of any such line with l_x. Since $T_x F$ is defined by the equation $y \wedge u(x) = 0$ for $y \in \mathbb{P}^5$, we get that $u(x) \in T_x F$, so $\pi(x)$ is the intersection of the lines l_x and $l_{u(x)}$, i.e. $\pi(x) = v$.

(2) If $v \in V^-$ we have $(v \wedge \wedge^2 V)^+ = v \wedge (V^+ \otimes V^-)$ and $(v \wedge \wedge^2 V)^- = v \wedge \wedge^2 V^+$.
   - We have $A^+ \cap (V^+ \otimes \wedge^2 V^-) = 0$ by Lemma 5.13, so there are no fixed points which arise from this case.
   - If $\alpha \in (v \wedge \wedge^2 V)^- \cap A^-$, there is $\tau \in \wedge^2 V^+$ such that $\alpha = v \wedge \tau$ and on the other hand there is $x \in \wedge^2 V^+$ such that $\alpha = f_1 \wedge x + f_2 \wedge u(x)$ by Lemma 5.13. Since f_1, f_2 form a basis of V^-, there is $\lambda \in \mathbb{C}$ such that $v = f_1 + \lambda f_2$. Comparing the two expressions we get $\tau = x$ and $u(x) = \lambda x$, i.e. λ is an eigenvalue of u and x is the corresponding eigenvector. Since u has 6 different eigenvalues, we obtain 6 isolated fixed points. □

REMARK 5.18. *When $v \in S$ the proof of Proposition 5.17 gives us more information : indeed, $\pi^{-1}(v) \cong \mathbb{P}((v \wedge \wedge^2 V)^- \cap A^-)$, hence $\dim(v \wedge \wedge^2 V)^- \cap A^- = 2$ if $v \in R$, 1 otherwise.*

When $v \in Q$ we always have that $\dim(v \wedge \wedge^2 V)^+ \cap A^+ = 1$, since by Lemma 5.13 the projection $A^+ \to V^+ \otimes \wedge^2 V^-$ is an isomorphism.

End of the proof of Proposition 5.16. From Remark 5.18 it follows that the only points v for which $\dim(v \wedge \wedge^2 V) \cap A = 3$ are among the 16 isolated singular points of S if they belong to Q too, which does not happen if Q is general enough. Hence we have $\dim(v \wedge \wedge^2 V) \cap A \leq 2$.

Moreover, we have seen that if $v \wedge w_1 \wedge w_2 \in A$ we can suppose $v \in V^+$ and we must have $\dim(v \wedge \wedge^2 V) \cap A = 2$. In such a case then $v \in Q \cap S$; we claim that v is then a singular point of the intersection, which cannot happen when Q is general enough.

Indeed, by Lemma 5.13 there is $x \in \wedge^2 V^+$ such that $v \wedge w_1 \wedge w_2 = v \wedge f_1 \wedge f_2 + \phi(v) + f_1 \wedge x + f_2 \wedge u(x)$ and by the proof of Proposition 5.17 we know that there are $z \in \wedge^2 V^+$ and $y_i \in V^+$ such that $\phi(v) = v \wedge z$, $x = v \wedge y_1$ and $u(x) = v \wedge y_2$. We can suppose that $w_i = w_i^+ + f_i$ with $w_i^+ \in V^+$ for $i = 1, 2$ up to a base change : we cannot have $W \subset V^+ \oplus \mathbb{C} f_i$ for $i = 1$ or 2, because in that case $v \wedge w_1 \wedge w_2 \in \wedge^3 V^+ \oplus (\mathbb{C} f_i \otimes \wedge^2 V^+)$ and $(\wedge^3 V^+ \oplus (\mathbb{C} f_i \otimes \wedge^2 V^+)) \cap A = 0$. After replacing we get

$$\begin{cases} v \wedge w_1^+ \wedge w_2^+ = v \wedge z \\ \\ v \wedge w_1^+ \wedge f_2 - v \wedge w_2^+ \wedge f_1 = v \wedge y_1 \wedge f_1 + v \wedge y_2 \wedge f_2 \end{cases} \tag{5.8.7}$$

From the second equation we get $w_1^+ - y_2 = kv$ and $w_2^+ + y_1 = hv$ with $k, h \in \mathbb{C}$. Hence we have $w_1^+ \wedge w_2^+ = y_1 \wedge y_2 + h y_1 \wedge v - k y_2 \wedge v$. Replacing in the first equation we then obtain $v \wedge (y_1 \wedge y_2 - z) = 0$, hence $\phi(v) = v \wedge z = v \wedge y_1 \wedge y_2 = 0$. This shows that y_1 and y_2 are in $T_v Q = \{y \in V^+ \mid y \wedge \phi(v) = 0\}$.

On the other hand, $T_v S$ is spanned by the two lines associated to $x = v \wedge y_1$ and $u(x) = v \wedge y_2$: both x and $u(x)$ satisfy $y \wedge u(x) = 0$ which is the equation defining $T_x F$, hence they span $\pi'(x) = T_v S$. Thus we get $T_v S = T_v Q$, i.e. v is a singular point of $Q \cap S$. This ends the proof of the smoothness of X_A when $A \in \mathbb{LG}(\wedge^3 V)^*$. $\square$

Let us now study the symplectic involution $\hat{\imath}$ induced on X_A (see Remark 5.10).

Proposition 5.19. *Let i be an involution on V such that $\dim V^+ = 4$ and let $A \in \mathbb{LG}(\wedge^3 V)^*$ be invariant for i ; then the induced symplectic involution $\hat{\imath}$ on X_A has 28 isolated fixed points and a fixed K3 surface.*

Proof. We know that $\mathrm{Fix}(\hat{\imath})$ has smooth symplectic components from Lemma 5.2 and Proposition 5.3 and on the other hand we have $f(\mathrm{Fix}(\hat{\imath})) \subset \mathrm{Fix}(i)$ which we have completely described.

If $Z \subset \mathrm{Fix}(\hat{\imath})$ is a surface it must be the double cover either of Q or of S and we know which is the ramification locus : it is given in the former case by $Q \cap W_A$ and in the latter by $S \cap W_A$.

From what we said above it is clear that $Q \cap W_A \cong Q \cap S$ and the double cover of a smooth quadric ramified along a quartic curve is a K3 surface. On the other hand let C be the trace of the quadric Q on the Kummer surface S ; then $S \cap W_A$ is the union of the 16 ordinary double points of S and of $C = Q \cap S$. In this case we have the following commutative diagram

$$\begin{tikzcd} T \arrow[r, "\varepsilon"] \arrow[d, "g"] & Z \arrow[d, "f_{|Z}"] \\ \Sigma \arrow[r, "\pi"] & S \end{tikzcd}$$

where π and ε are respectively the blow-ups of S and Z in $p_1, \ldots, p_{16}$ and in $f^{-1}(p_1), \ldots, f^{-1}(p_{16})$ and g is the double cover of Σ ramified along $E_1, \ldots, E_{16}$ and $\pi^{-1}(C) \cong C$. Let D be a divisor on Σ such that $2D = C$ and F_i be the exceptional divisor on T corresponding to $f^{-1}(p_i)$; we have that $K_T = g^* D + \sum F_i$ and also $K_T = \varepsilon^* K_Z + \sum F_i$, hence K_Z cannot be trivial and Z is neither K3 or abelian.

From what we said it follows that there cannot be abelian surfaces inside $\mathrm{Fix}(\hat{\imath})$ and that the symplectic involution is of the third type described by Theorem 5.5 and it must fix 28 isolated points and a K3 surface. We have already seen that the K3 surface arises as the double cover of Q ramified along the quartic curve $Q \cap S$. Moreover the 6 isolated fixed points on Y_A gives 12 fixed points for $\hat{\imath}$. Finally the 16 points $f^{-1}(p_i)$ which are the fibers of the 16 ordinary double points of S are fixed too, giving us all the 28 isolated points we expected. $\square$

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Stabilité des images inverses des fibrés tangents et involutions des variétés symplectiques

Résumé : Dans cette thèse j'ai travaillé sur deux problèmes différents dans le domaine de la Géométrie Algébrique.

La première partie de cette thèse consiste dans l'étude de la stabilité des images inverses du fibré tangent de l'espace projectif sur des variétés projectives. La stabilité de ces fibrés est équivalente à celle du noyau du morphisme d'évaluation M_L associé à un fibré en droites L engendré par ses sections globales. On obtient un résultat optimal dans le cas des courbes projectives et ensuite on utilise ce résultat pour en déduire la stabilité dans le cas des quelques surfaces projectives, notamment K3 et abéliennes.

Un second problème que nous abordons est l'étude du lieu fixe d'une involution symplectique d'une variété irréductible holomorphe symplectique de dimension 4 telle que $b_2 = 23$. On montre qu'il y a seulement trois cas possibles pour le nombre des points fixes isolés et des surfaces K3 fixées. On conjecture que seulement un cas soit possible, celui avec 28 points fixes isolés et une surface K3 fixée, et qu'une telle involution ne fixe jamais une surface abélienne. On vérifie cette conjecture dans quelques exemples.

Mots-clés : fibré vectoriel, fibré tangent, stabilité sur une courbe projective, stabilité sur une surface projective, variété holomorphe symplectique, automorphisme symplectique, involution symplectique, surface K3.

Stability of inverse images of tangent bundles and involutions of symplectic manifolds

Abstract : This thesis consists of two independent parts about two different problems in Algebraic Geometry.

In the first part we study the stability of inverse images of the tangent bundle of the projective space over projective varieties. The stability of these bundles turns out to be equivalent to the stability of the kernel of the evaluation map M_L of a line bundle L generated by its global sections. We obtain an optimal result in the case of projective curves and then we apply it to get the stability in the case of some projective surfaces, such as K3 and abelian surfaces.

The second problem we deal with is the study of fixed points of a symplectic involution over an irreducible holomorphic symplectic manifold of dimension 4 such that $b_2 = 23$. We show that there are only 3 possibilities for the number of fixed points and of fixed K3 surfaces. We conjecture that only one case can actually occurr, the one with 28 isolated fixed points and 1 fixed K3 surface, and that such an involution can never fix an abelian surface. We provide evidences for the conjecture by verifying it in some examples.

Keywords : vector bundle, tangent bundle, stability over a projective curve, stability over a projective surface, holomorphic symplectic manifold, symplectic automorphism, symplectic involution, K3 surface.

AMS Classification : 14H60, 14J50, 14J60, 32J27